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MATHEMATICS DEPARTMENT

THE

MATHEMATICAL GAZETTE

EDITED BY

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THE MATHEMATICAL ASSOCIATION.

THE Annual Meeting of the Mathematical Association was held at the Institute of Education on 5th-6th January, 1933.

On Thursday, 5th January, the proceedings opened at 2.15 p.m. with the transaction of business; the President, Professor G. N. Watson, was in the chair. The Report of the Council * for the year 1932 was presented and adopted, and the Treasurer's statement of the financial position received. The Officers and Council were re-elected for 1933, with the exception of Miss E. Wise and Professor G. B. Jeffery, who retired and were not eligible for re-election. Miss G. K. Stanley and Mr. F. C. Boon were elected in their places.

Professor Watson then delivered his Presidential address—*The Marquis and the Land Agent; a tale of the eighteenth century.*† This was followed, after an interval, by Dr. W. M. Smart's lecture on *The Precession of the Equinoxes.*

On Friday, 6th January, at 10 a.m., Mr. F. C. Boon gave a paper on *Sidetracks in Elementary Mathematics,*§ followed by Mr. A. Robson on *Parametric Equations in Elementary Analytical Geometry*‡ and Mr. A. W. Siddons on *Methods of Learning Geometrical Theorems,*‡ opening a discussion on this subject.

The afternoon meeting began with a discussion on *The Study of Statistics in a School Course,*‡ opened by Mr. F. Sandon, followed by Mr. C. O. Tuckey, Dr. J. Wishart and Mr. R. M. Wright. Mr. H. E. Piggott gave a paper on *Relative Velocity,*‡ and the concluding item was a lecture on *Lewis Carroll—Mathematician* ‡ by the Rev. D. B. Eperson.

The usual votes of thanks brought the proceedings to a close.

A Publishers' Exhibition was open during the two days.

* Pp. 2-4. † Pp. 5-17.

§ Pp. 18-25. ‡ To be published later.

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REPORT OF THE COUNCIL FOR THE YEAR 1932.

DURING the year 1932, 88 new members have been admitted to the Association. The number of members now on the Roll is 1298, of whom 4 are honorary members, 81 are life members by composition, 13 are life members under the old rule and 1200 are ordinary members.

The Council regrets to have to record the deaths of the following members of the Association :—Mr. W. A. Barclay, the Rev. C. H. Condell, Miss J. S. McLellan, Mr. T. V. Philpott, and Professor H. J. Priestley of the University of Queensland. An obituary notice of Professor Priestley appeared in the December issue of the *Gazette*.

The Teaching Committees.

The General Teaching Committee has had under consideration the Additional Mathematics of the Northern School Certificate Examination. This matter was referred to the Committee as the result of a discussion at the Annual General Meeting of the Association in 1932. The Committee drew up a draft syllabus on the lines suggested by the Manchester Branch and was prepared to suggest to the examining body that a credit might be given on some such syllabus, as an alternative to that now given in Mechanics. It was found, however, that the Branches concerned were not in agreement, and the matter was dropped.

A Report on the Teaching of Arithmetic in Schools was sent to all members of the Association with the February issue of the *Gazette*.

During the past year the Boys' Schools Committee has met regularly to prepare a report on the Teaching of Algebra in Schools to accompany the Arithmetic Report. It has also had under consideration the teaching of mathematics in Central Schools.

The Girls' Schools Committee has been considering entrance and scholarship tests for admission to Secondary Schools. Investigations have led to the conclusion that the problem of testing general ability by means of Arithmetic is a fruitful field for research on a larger scale than the Committee is able to undertake. The Committee is indebted to Miss Dale and to Mr. B. C. Wallis for very interesting and informing talks, and to Local Education Authorities, who supplied Scholarship Examination Papers and reports on them when available. Miss E. Wise resigned her office as Secretary of the Committee on being appointed Head Mistress of the High School, Skipton, and her place is now taken by Miss E. M. Read, whose address is 50 Bunbury Road, Northfield, Birmingham.

The Branches.

The Council has framed the following new Regulations : (i) A Branches Committee shall be formed consisting of the representatives of the Branches on the Council, together with the secretaries of those Branches not represented on the Council, and it shall appoint its own chairman and secretary. Its duties shall be to further the

exchange of information among the Branches, to report to the Council any matters of which that body should be cognisant, and in general to encourage and further the activities of the Branches. It shall also consider proposals for the formation of new Branches and report on them to the Council. (ii) Any Committee or Branch of the Association desiring to make a representation to an examining body or authority shall either state in making such representation that they are speaking for that Committee or Branch only, or, if they wish for the support of the whole Association, shall refer the matter to the General Teaching Committee. This Committee may either co-opt members of the above-mentioned Committee or Branch to discuss the matter with them, or may arrange for a joint meeting for the purpose. The representation, if agreed upon, shall be forwarded by the Chairman of the General Teaching Committee to the proper body or authority.

During the year seven meetings of the London Branch have been held, including a special visit to a display of statistical and accounting machinery. The Members' Topics meeting was again very successful. An innovation was a meeting at which the annual business was transacted, followed by a lantern lecture by Mr. A. Romney Green on "Some Interesting Practical Problems in Elementary Geometry". A successful Summer meeting was held at the Science Museum, where much trouble was taken by the authorities to show the collection of Calculating Machines. The membership of the Branch shows a slight increase to 187 members and 111 associates.

The Liverpool Branch has now 17 members and 52 associates. It has held four meetings during the year. A paper by Mr. A. W. Siddons on "The First Two Years' Course in Geometry" aroused great interest and drew an attendance of about a hundred.

Four meetings of the Midland Branch have been held during the year 1931-32, at which the average attendance was about 40. The number of members (28) and of associates (40) shows an increase on last year.

The North Eastern Branch has held four meetings during the year, one being in conjunction with the Sectional Conferences of University and Secondary School Teachers arranged by the University of Durham Schools Examination Board. The numbers attached to the branch are about 45 members and 10 associates.

The Yorkshire Branch (48 members and 115 associates) has held three ordinary meetings. Papers were read on "What is non-Euclidean Geometry?", "A Problem in Vibrations", "Modern Dynamics", and "Original Methods in Elementary Mathematics". A special meeting was called to consider the proposals of the Manchester Branch as to "Additional Mathematics". The annual dinner was held in the Spring term, Professor P. J. Daniell being the guest of the Branch.

The Southampton Branch has held three meetings at which addresses were given by Professor L. N. G. Filon on "Some Fallacies in the Teaching of Newtonian Mechanics"; by Mr. C. V.

Durell on "The Teaching to Schoolboys of Negative Numbers"; and by Brigadier Winterbotham on "The Geodesy of the Empire".

The Cardiff Branch (22 members and 22 associates) has held four well-attended meetings, and four meetings have been arranged for the new session also.

The Bristol Branch (12 members and 26 associates) has held four meetings at which papers have been read by Mr. A. H. Russell, Miss G. Yonge, Mr. G. J. B. Westcott and Dr. Bertha Swirles.

The Mathematical Gazette.

The issue for December, 1932, brings Vol. XVI to an end. It is intended to adhere in future to the arrangement that five numbers are to form one annual volume. An index to volumes I-XV is nearing completion and will probably be ready for distribution early in 1933.

Mr. Broadbent has now completed his second year of office and the Council desires to express its continued appreciation of the admirable manner in which his work as Editor has been done.

The Library.

The Library remains with Professor Neville, and the thanks of the Association are due to him and to Mr. Beames for their work during the past year.

The Problem Bureau.

The Problem Bureau continues to function, and it is hoped that it gives complete satisfaction to all applicants. Many members of the Association are deeply indebted to Mr. A. S. Gosset Tanner and to those who cooperate with him, and would like to express their grateful appreciation of the help which they have received.

Retirement from the Council.

The Council desires to express its thanks and those of the Association to Miss E. Wise and Professor G. B. Jeffery, who now retire from the Council, for the services which they have rendered to the Association during their tenure of office as members of the Council since 1929 and 1928 respectively.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. The names of those sending the questions will not be published.

THE MARQUIS AND THE LAND-AGENT; A TALE OF THE EIGHTEENTH CENTURY.

Presidential Address to the Mathematical Association, 1933.

By G. N. Watson, Sc.D., F.R.S.

There is a well-known story about the late Archbishop Temple, that he once had to listen to a sermon by a youthful and inexperienced clergyman, and to dine with him afterwards; the young man, by way of making conversation during the meal, ventured to remark, "I think, my lord, that I chose a good text for my sermon". Instantly there came the grim reply, "There was nothing wrong with the text". It may be that the consequence of my having selected a seductive title which does not possess a very close connection with the actual subject of my address will be that, when we adjourn presently, I may get the impression that my audience has consisted entirely of archbishops.

It seems to me that, whenever a Pure Mathematician is called upon to deliver a lecture of a more or less general nature outside the walls of his usual lecture room, the necessity of choosing a topic of general interest at once confronts him with a difficult problem; my choice of a title has been dictated by my desire to ensure that my address should not be entirely devoid of interest. An Applied Mathematician, of course, labours under no similar difficulty; it is always open to him to be irrelevant and to choose as his theme (if I may borrow a phrase of one of my predecessors) "what has been humorously called the real world, the world of physics and sensation, of sight and hearing, heat and cold, earthquakes and eclipses; and earthquakes and eclipses are plainly not constituents of the world of mathematics". It is, however, extremely difficult to give an interesting oral exposition of any modern developments in Pure Mathematics, and I personally have not the courage to attempt to do so; I can imagine few ways of rapidly emptying this room which would be more effective than an attempt on my part to expound the work on Singular Moduli which has occupied all my spare time during the last couple of years. I consider, too, that subjects of a pedagogical character are equally barred to me; I agree with Professor Hardy when he said on a similar occasion some years ago that he considered such questions to be of secondary importance.

Like many of the problems which confront the Pure Mathematician, the particular problem which I am considering always has one method of solution available; it is not in my opinion a particularly good solution, but I think that it is a perfectly valid procedure; and I have chosen the path of least resistance in adopting it, and therefore I have selected a topic of an historical nature; I propose to give you an account of some of the comparatively elementary work on arcs of ellipses and other curves which seems to me to have led up to the idea, familiar to you all, which came simultaneously to Abel and Jacobi 105 years ago, of inverting

an elliptic integral, and so laying the foundations of the theory of elliptic functions and doubly periodic functions generally. It may be a little rash for me to use the word historical; a distinguished historian has described the scope of his subject by saying, "Modern history began when Abraham came out of Ur of the Chaldees". I do not propose to go so far back as Abraham; but I would begin by referring you to I Kings vii. 23, where we are told that Solomon "made a molten sea, ten cubits from the one brim to the other; it was round all about, and his height was five cubits; and a line of thirty cubits did compass it round about". It has been suggested, I think rather too ingeniously, that this passage does not bear the obvious interpretation that a comparatively primitive people regarded π as being equal to 3, but that the word round was used in a vague sense and that the vessel in question was elliptical, its major axis being ten cubits and its minor axis being about 9·53 cubits.

The earliest indubitable reference to the length of an elliptic arc that I have been able to discover occurs in Kepler's *Astronomia nova* (Prag, 1609); his discovery, announced in the same work, that the orbit of the planet Mars is an ellipse naturally drew attention to the length of the perimeter of an ellipse. In modern notation, Kepler's theorem is that the perimeter of an ellipse with semi-axes a and b exceeds $2\pi\sqrt{(ab)}$; his proof consists of an application of the theorem that, of figures which enclose the same area, the circle has the least perimeter; and the perimeter of the circle which has the same area as the ellipse is $2\pi\sqrt{(ab)}$. This theorem is followed by the naïf remark that the perimeter of the ellipse is nearly equal to the arithmetic mean of the circumferences of the major and minor auxiliary circles, for the somewhat slender reason that both the perimeter and the arithmetic mean exceed the geometric mean of the circumferences. Things which are greater than the same thing are nearly equal to each other!

Passing over half a century, I come to 1659, in which year Pascal published his essay on *Dimensions des lignes courbes de toutes les Roulettes*, and Wallis published two *Tracts*; Pascal's work and the second tract of Wallis both contain the theorem that the span of the curtate cycloid traced by a point distant b from the centre of a rolling circle of radius a is equal to the semi-perimeter of an ellipse whose semi-axes are $2(b+a)$ and $2(b-a)$. The proof given by Wallis is apparently due to Wren, and I am convinced that Wren discovered both the theorem and his proof of it before July 1658, while I am doubtful whether Pascal knew of the theorem until about the time that he completed his work on Roulettes in October 1658. I regard Wren's proof as elegant and direct, but Pascal's reasoning seems to me to be obscure, like other writings of his that I have read. The second tract of Wallis is also remarkable for containing an expression for the differential element of arc of an ellipse in terms of the abscissa, and the construction of a curve about which he says "the figure so constructed is no more capable of quadrature than an ellipse of rectification".

By applying the binomial theorem to the differential element of arc and integrating the series so obtained term by term, Newton obtained an expression for the length of a quadrant of an ellipse as the sum of two infinite series in 1676; the second series is convergent only when the eccentricity does not exceed $\frac{1}{2}$.

The series for the length of a quadrant in powers of the eccentricity, namely

$$a\int_0^{\frac{1}{2}\pi}\sqrt{(1-\epsilon^2\sin^2\theta)}\,d\theta$$
$$=\tfrac{1}{2}\pi a\left[1-\frac{1}{2.2}\epsilon^2-\frac{1.1.3}{2.2.4.4}\epsilon^4-\frac{1.1.3.3.5}{2.2.4.4.6.6}\epsilon^6-\ldots\right],$$

seems to have been discovered by Maclaurin in 1742, and it is probably the best known formula of its kind.

The source of the main topic of my address is not, however, to be found in the discoveries which I have just described. It consists of an observation made by Jacob Bernoulli in 1691, namely that, in the expression for the arc of the parabolic spiral

$$(a-r)^2=2ab\theta$$

as an integral
$$s=\int\sqrt{\left\{1+\frac{r^2(a-r)^2}{a^2b^2}\right\}}\,dr,$$

the integrand is an *even function* of $\frac{1}{2}a-r$, so that $\int_{\frac{1}{2}a-c}^{\frac{1}{2}a}=\int_{\frac{1}{2}a}^{\frac{1}{2}a+c}$, that is to say, the arc of the curve joining the points for which r has the values $\frac{1}{2}a-c$ and $\frac{1}{2}a$ is equal to the arc of the curve joining the points for which r has the values $\frac{1}{2}a$ and $\frac{1}{2}a+c$, *although these arcs are not congruent*.

This observation, which nowadays would be regarded as almost a truism, inspired an Italian mathematician Giulio Carlo di Fagnano, to carry out the researches on arcs of equal lengths which resulted in his being loaded with honours in his lifetime and which constitute his claim to mathematical immortality.

Before I proceed to describe some of Fagnano's discoveries, I extract a few biographical details from Gambioli's account of his life contained in the third of the sumptuous volumes of Fagnano's *Opere matematiche* which were published in 1911-12.

Fagnano's family derived from the ancient lords of the Castle of Fagnano between Bologna and Imola; to this family belonged Honorius II, elected pope in 1124. Later the family flourished in Bologna under the surname of Toschi; about 1341 they settled in Senigallia, a little town on the shores of the Adriatic, about half-way between Pesaro and Ancona, now a sea-bathing resort of 12,000 inhabitants.

Giulio Carlo, son of Francesco Fagnano and his wife Camilla Bartoli, was born at Senigallia on 26th September 1682, and died there on 18th May 1766. From the age of ten he enjoyed poetry and composed sonnets; later he published a philosophical poem on the resurrection of the dead. At the age of fourteen he entered

the Clementine College at Rome, where he studied for three years; he quickly attained distinction in the College, and completed his philosophy course in two years and the theological course in one year. He abandoned the study of the Aristotelian philosophy for the works of Gassendi and Descartes; he got into communication with Malebranche and gave him a new explanation of the mystery of transubstantiation, receiving an alternative explanation from him in return. He proceeded to the philosophy of Leibniz and Newton, and derived much profit from studying the famous work of Malebranche, *La recherche de la verité*. While a student he held mathematics in contempt, in spite of advice to study the subject; but later he decided to abandon philosophy and to devote himself entirely to mathematics. According to Gambioli, he was entirely self-taught, so far as mathematics were concerned, and Gambioli emphasizes the mathematical ability which he must have possessed in being able to study the subject so effectively after attaining years of maturity, in a small town like Senigallia, without a library and remote from any centre of learning.

In 1743 he was consulted with regard to the safety of St. Peter's, which was threatened with collapse; and, as a reward for his services, Benedict XIV commissioned the publication of his collected works, though the two volumes composing them did not actually appear until 1750. He had been created a Count by Louis XV in 1721, and in 1745 the Pope created him a Marquis; when the Roman nobility was reconstituted in the following year, he was included in the 101 noble families. He was also created Marquis of Sant' Onorio by Charles III, King of the two Sicilies. He was nominated to the Berlin Academy in 1750, and he sent the Academy a copy of his *Produzioni matematiche* which reached Euler's hands on 23rd December 1751, a day described by Jacobi as "Der Geburtstag der elliptischen Funktionen".

He married Francesca Conciatti of Sassoferrato, by whom he had six sons and six daughters; she died in 1726, and but four of his children survived him; one of the sons was a mathematician, but he was less able than his father; the family is now extinct.

Gambioli gives a lively account of his personality; he describes him as cheerful and lively in conversation; and relates, as an instance of his equanimity, a story of how, one evening, he was driven to shooting two robbers and, meeting some friends shortly afterwards, he conversed with them as though nothing had happened.

He died on 18th May 1766, and is buried in the church of Santa Maria Maddalena at Senigallia; the municipality of Senigallia possesses his portrait, representing him with a diagram of a lemniscate in his hand; below is an elaborate inscription describing him as "Julius Carolus de Tuschis de Fagnano, S. Honorii Marchio, in Ordine Constantin. S. Georgii Marchiae Prior, Patricius Romanus et Senogalliensis". He is described in a similar manner on the title-page of his *Produzioni matematiche* and below his titles is a diagram of a lemniscate with the words "Multifariam divisa atque dimensa. Deo veritatis gloria".

I come now to a brief description of Fagnano's work; his investigations on the geometrical theory of proportion, which occupy a large part of the first volume of his *Produzioni matematiche*, are, of course, irrelevant, and I do not hesitate to say that I naturally consider them rather dull reading. I pass over, also, his work on the rectification of parabolas of higher order which was a preliminary to his more famous discoveries published from 1714 onwards, and connected, as might be anticipated from some hints which I have given, with the rectification of the lemniscate.

Taking the equation of the lemniscate in the usual form

$$r^2 = 2a^2 \cos 2\theta,$$

and writing $\cos 2\theta = t, \quad r = a\sqrt{(2t)},$

we obtain the parametric representation

$$x = a\sqrt{(t+t^2)}, \quad y = a\sqrt{(t-t^2)},$$

whence the expression for the arc in the form of an integral

$$s = \frac{a}{\sqrt{2}} \int \frac{dt}{\sqrt{\{t(1-t^2)\}}}$$

is immediately derivable.

Fagnano's first discovery was that, if two parameters t and z are connected by the relation

$$(t+1)(z+1) = 2,$$

then $$\frac{dt}{\sqrt{\{t(1-t^2)\}}} = -\frac{dz}{\sqrt{\{z(1-z^2)\}}},$$

and hence $$\int_0^z \frac{du}{\sqrt{\{u(1-u^2)\}}} = \int_t^1 \frac{du}{\sqrt{\{u(1-u^2)\}}}.$$

In modern terminology, he had discovered a homographic transformation under which the integrand is invariant, its singularities being permuted among themselves.

I remark that the points with parameters 0 and 1 are the centre C and the end of the transverse axis A; so that, using an obvious notation, we see that the arcs Cz and tA are equal. If now we take $t = z$, it follows from the homographic relation that each is equal to $\sqrt{2}-1$, so that *the point with parameter* $\sqrt{2}-1$ *bisects the arc forming the positive quadrant of the lemniscate.*

To describe Fagnano's second discovery, it is convenient to take a slightly different parametric representation

$$x = az\sqrt{(1+z^2)}, \quad y = az\sqrt{(1-z^2)},$$

by writing z^2 for t in the former representation.

The formula for the arc is now

$$\text{Arc } Cz = a\sqrt{2} \int_0^z \frac{du}{\sqrt{(1-u^4)}},$$

and the first theorem becomes:

If $\zeta^2 = \dfrac{1-z^2}{1+z^2}$ *then* $\displaystyle\int_0^z = \int_\zeta^1.$

Now consider the transformation

$$\zeta=\frac{1-w^2}{1+w^2},$$

so that

$$\left(\frac{1-w^2}{1+w^2}\right)^2=\frac{1-z^2}{1+z^2}.$$

It is not very difficult to verify that this may be written in the equivalent forms

$$z=\frac{w\sqrt{2}}{\sqrt{(1+w^4)}},\quad \frac{dz}{\sqrt{(1-z^4)}}=\sqrt{2}\frac{dw}{\sqrt{(1+w^4)}}.$$

The form of this relation is the key to Fagnano's subsequent work; the method by which he was led to discover the relation is not clear, and I think that it must be regarded as one of those happy chances which, as Legendre once remarked in another connection, occur only to those who know how to produce them.

Unlike the first transformation, the second does not give an invariant integral; but, if we take

$$w=\frac{v\sqrt{2}}{\sqrt{(1-v^4)}},$$

we get

$$\frac{dw}{\sqrt{(1+w^4)}}=\sqrt{2}\frac{dv}{\sqrt{(1-v^4)}},$$

and hence

$$\frac{dz}{\sqrt{(1-z^4)}}=2\frac{dv}{\sqrt{(1-v^4)}}.$$

This is a rudimentary form of what is now known as complex multiplication; the relation between z and w is of the same form as the relation between $w\eta$ and $v\eta$, where η is a fourth root of -1. Whether Fagnano discovered the transformation in this way or not, I do not know; it is by no means impossible that this was his method, because he had made an intensive study of the technique of the algebra of complex numbers, and, in fact, he was the discoverer of the well known formula

$$\pi=2i\log\frac{1-i}{1+i}.$$

The last differential equation may be written in the equivalent forms

$$\text{Arc } Cz=2\text{ Arc } Cv,\quad v=\frac{z}{\sqrt{[\{1+\sqrt{(1-z^2)}\}\{1+\sqrt{(1+z^2)}\}]}}.$$

Take now a new parameter u, defined by the equation

$$u^2=\frac{1-v^2}{1+v^2},$$

and then, by the first result,

$$\text{Arc } Cv=\text{Arc } uA,\qquad \text{Arc } Cz=2\text{ Arc } uA,$$

where the equation connecting u and z can be worked out without much difficulty. If, in this equation, we take $u=z$, the positive

quadrant is trisected at v and u; and then the equation for z reduces to

$$z^8+6z^4-3=0,$$

whence $$z=\sqrt[4]{(2\sqrt{3}-3)},$$

so that the point with parameter $\sqrt[4]{(2\sqrt{3}-3)}$ *is one of the points of trisection of the arc forming the positive quadrant of the lemniscate.*

The problem of the quinquesection of the quadrant was attacked by Fagnano in a similar manner. It is necessary to take four parameters, t, z, v, u, representing points on the lemniscate such that

$$\text{Arc } Ct=2 \text{ Arc } Cz, \quad \text{Arc } Cz=2 \text{ Arc } Cv, \quad \text{Arc } Cv=\text{Arc } uA,$$

so that consecutive parameters are connected either by the first or by the second of Fagnano's relations, whence an algebraic equation equivalent to the formula

$$\text{Arc } Ct=4 \text{ Arc } uA,$$

connecting t and u, is obtainable.

If, in this equation, we take $t=u$, we get

$$\text{Arc } Ct=\tfrac{4}{5} \text{ Arc } CA,$$

and the equation for t (which was not worked out by Fagnano) reduces to

$$t^{24}+50t^{20}-125t^{16}+300t^{12}-105t^8-62t^4+5=0.$$

When we discard from this equation factors which may be proved to be irrelevant, we find that

$$t^8-(12\sqrt{5}-26)t^4+9-4\sqrt{5}=0,$$

and hence the problem of the quinquesection of the quadrantal arc of the lemniscate is reduced ultimately to the solution of quadratic equations.

As I said, Fagnano did not work out this result to the bitter end; but he clearly realized that the work was possible, for he went so far as to state the general theorem that, if m is any integer, it is possible to find, *algebraically*, a point with parameter z such that the arc Cz is any one of the following three fractions of the length of the quadrant of the lemniscate:

$$\frac{1}{2^m}, \quad \frac{1}{3.2^m}, \quad \frac{1}{5.2^m};$$

and then comes the remark which I think must have been uttered with a pride that was fully justifiable: "E questa è una nuova, e singolare proprietà della mia curva".

I now temporarily leave the work of Fagnano and turn to that of John Landen. There is a short biography of Landen in the *D.N.B.* He was born near Peterborough on 23rd January, 1719, and brought up to the business of a surveyor. He acted as land-agent to William Wentworth, Earl Fitzwilliam, from 1762 to 1788. He published a small volume not very happily entitled *Mathematical Lucubrations* in 1755, about which I shall have something

to say in a moment. His most important work from my present point of view is contained in some later papers published in *Phil. Trans. Roy. Soc.* He had been elected a Fellow of that Society in 1766. In addition to his work on elliptic arcs he investigated certain dynamical problems, about which the *D.N.B.* merely states that his results differed from those of Euler and d'Alembert. He is not the only writer on dynamical problems whose results are different from those of recognised authorities; a similar instance is to be found in the *Quarterly J. of Math.*, 48, published as recently as 1920. Landen was not to be convinced of the incorrectness of his views and defended them in a volume of memoirs which he prepared for press while suffering from a painful disease; a copy of the volume was placed in his hands the day before his death, which occurred on 15th January 1790.

The *Mathematical Lucubrations* contain one passage which is relevant here. Landen gives the following definitions: "e denotes one fourth of the periphery of an Ellipsis whose semiaxes are $\sqrt{2}$ and 1. f denotes one fourth of the periphery of a Circle whose radius is 1". He then makes a number of extremely odd-looking statements of which I quote two as typical:

$$\int_0^1 \frac{x^{-\frac{1}{2}}\,dx}{\sqrt{(1-x^2)}} = e + \sqrt{(e^2-2f)}, \quad \int_0^1 \frac{x^{\frac{1}{2}}\,dx}{(1-x^2)^{\frac{1}{2}}} = \frac{e-\sqrt{(e^2-2f)}}{\sqrt{2}}.$$

The clue to these theorems is found by taking Maclaurin's formula (already quoted) in the form

$$e = \int_0^{\frac{1}{2}\pi} \sqrt{(2-\sin^2\theta)}\,d\theta,$$

and making the substitutions

$$\sin\theta = \sqrt{2}\,.\,\sin\phi, \quad \cos^2 2\phi = y,$$

whence it follows that

$$e = \frac{1}{4}\left[\int_0^1 y^{-\frac{3}{2}}(1-y)^{-\frac{1}{2}}\,dy + \int_0^1 y^{-\frac{1}{2}}(1-y)^{-\frac{1}{2}}\,dy\right].$$

Landen had previously obtained a modification of the well-known product given by Wallis for π in the form

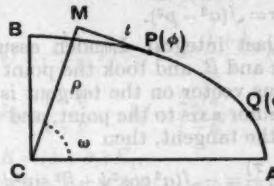
$$\int_0^1 y^{-\frac{3}{2}}(1-y)^{-\frac{1}{2}}\,dy \times \int_0^1 y^{-\frac{1}{2}}(1-y)^{-\frac{1}{2}}\,dy = 4\pi$$

by a very ingenious piece of analysis (I remark that the truth of Landen's result is evident when the integrals are expressed in terms of gamma functions), and it is an immediate consequence of the last two equations that

$$\sqrt{(e^2-2f)} = \frac{1}{4}\left[\int_0^1 y^{-\frac{3}{2}}(1-y)^{-\frac{1}{2}}\,dy - \int_0^1 y^{-\frac{1}{2}}(1-y)^{-\frac{1}{2}}\,dy\right].$$

The whole set of Landen's formulae may now be obtained by quite elementary transformations.

I take next a very beautiful theorem, first discovered in an algebraical form by Fagnano, given a geometrical interpretation by Euler, and then modified and developed by Landen. The investigation which I give is due to Legendre; it seems to me much more elegant than the earlier work on the theorem. (I remark in parenthesis that it has been my experience—I do not know to what extent my impressions are shared with others—that the technique of eighteenth century mathematicians in dealing with problems of pure calculus was considerably superior to their technique in analytical geometry, and that frequently the algebraical work which they found necessary in solving problems of analytical geometry can only be described as clumsy when compared with their other work.)



To return to the theorem, a point P with eccentric angle $\frac{1}{2}\pi-\phi$ is taken on an ellipse, M is the foot of the central perpendicular on the tangent at P, p and ω are the polar coordinates of M, Q is the point with eccentric angle $\frac{1}{2}\pi-\omega$, and finally t and s are defined as the length MP and the arc BP.

The coordinates of P being (x, y), the truth of the equations

$$\frac{dx}{ds}=\sin\omega, \quad \frac{dy}{ds}=-\cos\omega,$$

$$p=x\cos\omega+y\sin\omega, \quad t=x\sin\omega-y\cos\omega=-\frac{dp}{d\omega}$$

is then fairly evident. Hence

$$\frac{d(s-t)}{d\omega}=\frac{(dx)^2+(dy)^2}{ds\,d\omega}-\frac{dt}{d\omega}=\sin\omega\frac{dx}{d\omega}-\cos\omega\frac{dy}{d\omega}-\frac{dt}{d\omega}=-p.$$

It then follows that

$$\left[s-t\right]_{\omega}^{\frac{1}{2}\pi}=-\int_{\omega}^{\frac{1}{2}\pi}p\,d\omega,$$

$$s-t=\int_{\omega}^{\frac{1}{2}\pi}p\,d\omega=\int_{\omega}^{\frac{1}{2}\pi}\sqrt{(a^2\cos^2\omega+b^2\sin^2\omega)}\,d\omega;$$

and, by Maclaurin's formula, the integral on the right is equal to QA. *Hence*

$$\text{Arc } BP-\text{Arc } QA=MP.$$

The coordinates of Q being called (x', y'), we have

$$x=\frac{a^2\cos\omega}{\sqrt{(a^2\cos^2\omega+b^2\sin^2\omega)}}, \quad x'=a\sin\omega,$$

and hence

$$x^2x'^2(a^2-b^2)-(x^2+x'^2)a^4+a^6=0,$$

a result which may be expressed much more elegantly in the form

$$\tan\phi\tan\omega=\frac{a}{b}.$$

Landen's most important discovery was a consequence of taking the corresponding formula for a hyperbola; the arc being now measured from the end of the major axis and ω being negative, we have

$$t-s=\int_{\omega}^{0}\sqrt{(a^2\cos^2\omega-b^2\sin^2\omega)}\,d\omega$$
$$=\int_0^{\tau}\sqrt{\left(\frac{(a^2-\tau^2)}{a^2+b^2-\tau^2}\right)}\,d\tau,$$

where $\tau=\sqrt{(a^2-p^2)}$.

To deal with the last integral, Landen assumed an ellipse of arbitrary semi-axes, α and β, and took the point of it for which the projection of the radius vector on the tangent is τ. If σ is the arc from the end of the minor axis to the point, and ψ the inclination of the perpendicular on the tangent, then

$$\frac{d(\sigma-\tau)}{d\psi}=-\sqrt{(\alpha^2\cos^2\psi+\beta^2\sin^2\psi)}.$$

Now τ is expressible as a function of ψ; and, by taking τ as independent variable in place of ψ, Landen found, with the help of some heavy algebra, that

$$2\frac{d\sigma}{d\tau}=1+\frac{\alpha^2+\beta^2-\tau^2}{\alpha^2-\beta^2}\left[1-\frac{2(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)^2}\tau^2+\frac{\tau^4}{(\alpha^2-\beta^2)^2}\right]^{-\frac{1}{2}}.$$

He then chose α and β in terms of a and b, so that the expression in [ ] factorized into

$$\left(1-\frac{\tau^2}{a^2}\right)\left(1-\frac{\tau^2}{a^2+b^2}\right)$$

the connection of which with the original integrand is evident. It turns out that this choice makes

$$\alpha,\ \beta=\frac{\pm a+\sqrt{(a^2+b^2)}}{2}.$$

It then follows that

$$2\frac{d\sigma}{d\tau}=1+\frac{(a^2-\tau^2)+(a^2+b^2-\tau^2)}{2\sqrt{\{(a^2-\tau^2)(a^2+b^2-\tau^2)\}}}$$
$$=1+\frac{d(t-s)}{2\,d\tau}+\frac{1}{2}\sqrt{\left(\frac{a^2+b^2-\tau^2}{a^2-\tau^2}\right)},$$

and hence, on integration,

$$s=t+2\tau-4\sigma+\sigma_1,$$

where σ_1 is an arc of an ellipse with semi-axes $\sqrt{(a^2+b^2)}$ and b.

Pairs of ellipses whose semi-axes are related in the manner of the two ellipses of this problem are said to be connected by Landen's transformation. In the hands of Legendre, the transformation became a most powerful method for computing elliptic integrals;

and the transformation made possible the theory of more general transformations, leading up to the theories of modular equations, complex multiplication, and singular moduli.

A few words are permissible about Legendre's applications. In his notation

$$F(\phi, k)=\int_0^{\phi}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}=\int_0^{\phi}\frac{d\phi}{\Delta(\phi, k)};$$

$$F(\tfrac{1}{2}\pi, k)=K,\quad F(\pi, k)=2K;\quad k'=\sqrt{(1-k^2)};$$

and the change of variable which effects Landen's transformation is

$$\sin\phi_1=\frac{(1+k')\sin\phi\cos\phi}{\Delta(\phi, k)}.$$

We then have

$$F(\phi, k)=\frac{1}{1+k'}F(\phi_1, k_1)=\frac{1+k_1}{2}F(\phi_1, k_1),$$

where

$$k_1=\frac{1-k'}{1+k'},\quad k_1'=\frac{2\sqrt{k'}}{1+k'},$$

and so

$$K=(1+k_1)K_1.$$

Legendre then got the idea of repeating the transformation so as to have a sequence or *scale*, $k_1, k_2, k_3, \ldots$, such that

$$k_{m+1}=\frac{1-k_m'}{1+k_m'}=\frac{k_m^2}{(1+k_m')^2},$$

so that $k_{m+1}<k_m^2$ and, by induction, $k_m<k^{2^m}$. Further

$$K=K_\infty\prod_{m=1}^{\infty}(1+k_m)=\frac{\pi}{2}\prod_{m=1}^{\infty}(1+k_m),$$

and the convergence of the sequence k_m to zero is most extraordinarily rapid. Even if k is as large as 0·99990, then

$$k_4=0{\cdot}00367\,06,\quad k_5=0{\cdot}00000\,34.$$

For some purposes it is better to follow Gauss and work with the semi-axes of ellipses rather than with the eccentricity in the manner of Legendre. If *

$$\sin\phi_1=\frac{(a+b)\sin\phi\cos\phi}{\sqrt{(a^2\cos^2\phi+b^2\sin^2\phi)}},$$

then

$$\frac{d\phi_1}{\sqrt{\{(a+b)^2\cos^2\phi_1+4ab\sin^2\phi_1)\}}}=\frac{d\phi}{\sqrt{(a^2\cos^2\phi+b^2\sin^2\phi)}},$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi_1}{\sqrt{(a_1^2\cos^2\phi_1+b_1^2\sin^2\phi_1)}}=\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(a^2\cos^2\phi+b^2\sin^2\phi)}},$$

where

$$a_1=\tfrac{1}{2}(a+b),\quad b_1=\sqrt{(ab)}.$$

Repetitions of the operation give a double sequence (a_m, b_m), whose members converge with similar rapidity to a common limit, called,

* Gauss actually made a different transformation, but one which led to the same result.

for obvious reasons, the arithmetic-geometric mean of a and b; if it is denoted by $\mathfrak{M}(a, b)$, then

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}} = \frac{\frac{1}{2}\pi}{\mathfrak{M}(a, b)}.$$

It is not irrelevant to quote here the investigation given in 1796 by Ivory (better known for his work on gravitating ellipsoids) on the perimeter of an ellipse, since it is connected with Maclaurin's formula by Landen's transformation.

If x denotes $(a-b)/(a+b)$, the perimeter is equal to

$$\begin{aligned}
&4\int_0^{\frac{1}{2}\pi} \sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}\,d\phi \\
&= 2(a+b)\int_0^{\frac{1}{2}\pi} \sqrt{\left\{1 + 2\frac{a-b}{a+b}\cos 2\phi + \left(\frac{a-b}{a+b}\right)^2\right\}}\,d\phi \\
&= 2(a+b)\int_0^{\frac{1}{2}\pi} \sqrt{\{(1+xe^{2i\phi})(1+xe^{-2i\phi})\}}\,d\phi \\
&= 2(a+b)\int_0^{\frac{1}{2}\pi} \left(1 + \tfrac{1}{2}xe^{2i\phi} - \frac{1.1}{2.4}x^2e^{4i\phi} - \frac{1.1.3}{2.4.6}x^3e^{6i\phi} - \dots\right) \\
&\qquad \times \left(1 + \tfrac{1}{2}xe^{-2i\phi} - \frac{1.1}{2.4}x^2e^{-4i\phi} - \frac{1.1.3}{2.4.6}x^3e^{-6i\phi} - \dots\right) d\phi \\
&= \pi(a+b)\left[1 + (\tfrac{1}{2})^2x^2 + \left(\frac{1.1}{2.4}\right)^2x^4 + \left(\frac{1.1.3}{2.4.6}\right)^2x^6 + \dots\right],
\end{aligned}$$

and this is Ivory's series.

To return to Gauss, it would be absurd for me to stop short with the description of the arithmetic-geometric mean without mentioning a much more noteworthy discovery of his, and one which is not so generally known. Although Gauss published little during his lifetime, he kept a fairly full diary for a number of years, and under the date 8th January 1797 is to be found the concept of "inverting" the integral which occurs in Fagnano's work on the lemniscate, thirty years before the inversions due to Abel and Jacobi.

As samples of the definitions and the simpler formulae discovered by Gauss I quote:

$$\phi = \int_x^1 \frac{dx}{\sqrt{(1-x^4)}}, \quad \tfrac{1}{2}\varpi = \int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = 1{\cdot}31102\,878\dots,$$

$$x = \operatorname{coslemn}\phi = \operatorname{cl}\phi,$$

$$\operatorname{sinlemn}\phi = \operatorname{sl}\phi = \operatorname{cl}(\tfrac{1}{2}\varpi - \phi),$$

$$\operatorname{sl}^2\phi = \frac{1-\operatorname{cl}^2\phi}{1+\operatorname{cl}^2\phi}.$$

The connection between the Gaussian functions and the Jacobian functions is expressed by the formulae

$$\operatorname{sl}\phi = \frac{1}{\sqrt{2}}\operatorname{sd}\left(\phi\sqrt{2}, \frac{1}{\sqrt{2}}\right), \quad \operatorname{cl}\phi = \operatorname{cn}\left(\phi\sqrt{2}, \frac{1}{\sqrt{2}}\right).$$

It is well known that the reluctance of Gauss to publish his discoveries was due to the rejection of his *Disquisitiones arithmeticae* by the French Academy, the rejection being accompanied by a

sneer which, as Rouse Ball has said, would have been unjustifiable even if the work had been as worthless as the referees believed. It is the irony of fate that, but for this sneer, the *Traité des fonctions elliptiques*, the work of a Frenchman, might have assumed a different and vastly more valuable form, and Legendre might have been spared the pain of realizing that many years of his life had been practically wasted, had the method of inversion come to be published when Legendre's age was fifty instead of seventy-six.

I have now completed my tale of the work of the eighteenth century; but, just as I began by quoting an empirical formula of a hundred years earlier, I shall end by quoting some empirical formulae of a hundred years later; the first of these formulae for the perimeter of an ellipse is due to Peano (1887), namely

$$\pi[a+b+\tfrac{1}{2}(\sqrt{a}-\sqrt{b})^2];$$

it contrasts rather curiously with the type of work for which he is best known. It is not to be expected that I should so restrain myself as to refrain from mentioning the Indian mathematician Ramanujan, and the last two formulae are due to him (1914):

$$\pi[3(a+b)-\sqrt{\{(a+3b)(3a+b)\}}],$$

$$\pi\left[a+b+\frac{3(a-b)^2}{10(a+b)+\sqrt{(a^2+14ab+b^2)}}\right].$$

I have proved that in these formulae the errors (which are zero in the limiting case of the circle) are in defect and that they increase when the eccentricity is increased, though they are extremely small even in the extreme case of the parabola. Ramanujan has mentioned that, for small values of the eccentricity ϵ, the errors are about

$$\pi a\epsilon^{12}/10\,48576, \quad 3\pi a\epsilon^{20}/6\,87194\,76736.$$

My final task, and a very pleasant one, is to express most cordially my gratitude to Professor Neville for having made the extremely happy suggestion of an exhibition of books on elliptic functions so as to give some indication of the growth of the subject during the last century. Many of the books come from his own library; and my appreciation of his idea and of the energy with which he has carried out the task of making the exhibition complete will be shared by all who see the collection. G. N. W.

[The exhibition consisted of 97 volumes, not counting a few duplicates, and included all but a very few of the existing books on elliptic functions. For this measure of completeness members of the Association are indebted to Dr. J. R. Airey, Dr. L. J. Comrie, and Mr. F. Robbins, and to the Libraries of the University of Birmingham; University College, London; University College, Bangor; and St. John's and Trinity Colleges, Cambridge, as well as to their own President and Librarian and the Editor of the *Gazette*. The Librarian wishes to record his appreciation of the courtesy and helpfulness with which his requests were met, not only by those who actually lent books, but also by others in commercial as well as in academic circles who found they had nothing to add by the time they were approached.

Before the collection is dispersed, the most careful bibliographical record will be compiled, and it is hoped that this will become of permanent value to librarians and scholars.]

SIDETRACKS IN ELEMENTARY MATHEMATICS.*

By F. C. Boon.

The remarks I have to offer are made with no little diffidence, but no great reluctance. I am diffident because I feel I can only offer a very humble contribution to your meeting; at the same time, recalling the aims of the Association and the purpose of this meeting, I am glad to bring up for discussion a subject which concerns our teaching in all branches of the subject and at all stages.

As good mathematicians you will expect me to begin by defining my terms. What do I mean by "Sidetracks"?

If you are a motorist, you probably press to your destination with all the speed and directness that your horse-power and an arterial road make possible.

If you have spent holidays in the Alps, you have found that the direct route along a glacier is interrupted by a welter of impassable crevasses, and that a point half a mile above you is only attainable after hours of zigzagging up the mountain side by paths that never point to your destination at all.

If you are a hiker and love the companionable English countryside, you take every opportunity of leaving the high road and following uncrowded ways where wild flowers bloom, birds sing and the path under arching trees is dappled with sunlight.

There are teachers who drive a class with the insistence of their personality and an uncompromising eye on a syllabus; there are others who circumvent their pupils' difficulties by contriving easy ways of approach; and there are those who are ready to break away from routine at any invitation.

May I dispense with definition and proceed to give examples of what one or other of us would regard as sidetracks? I can only touch my subject at a few points. I shall endeavour not to stress my own opinion but ask you to amplify my remarks out of your own experience.

The beginner in Geometry is faced with a formidable array of hefty words: *diagonal*, *isosceles*, *parallelogram*, etc. Shall we be content if the meaning and spelling are learnt, or shall we follow, from time to time, a digression into etymology?

Consider the word *diagonal*. It contains the root *gon*, which appears also in *polygon* and, later, in *trigonometry*, which means *angle* and is connected with the Greek γόνυ and Latin *genu*, a knee; *dia* the pupil finds again in *diameter* with the same meaning of *through*; *meter* he knows already in *gas meter*, *speedometer* and *metre*.

You may be asked by some unnaturally curious pupil why the symbol x^2 should be called *x square*. You will point out that other nations use the same phrase; the Germans, *x quadrat*; the French, *x carré*. These words, derived from the Latin *quadratus*, can be connected with *square*, and through *escadron*, with *squadron*, with *quad-*

* A paper read at the Annual Meeting of the Mathematical Association, January 6, 1933.

ratic, quadrille, etc. This will lead to the information that Latin was the Lingua Franca of medieval culture, enabling Newton to correspond with Leibnitz more easily than Einstein can with Eddington. The use of the phrase will suggest that arithmetic and algebra borrowed the language of geometry because geometrical theory was developed first and geometrical diagrams illustrated arithmetical theory. The Greeks of the classical period used τετράγωνος ἀριθμός, a *square*

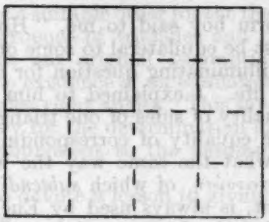


FIG. 1.

number, and illustrated it by a unit square with gnomons attached, showing that S_n of $1+3+5+\ldots=n^2$. Thus the phrase goes back to the early days of mathematical study.

Quite recently, when I said something of this sort to a class, I was asked when algebra was introduced. This led to another digression in which we followed the travels of Greek mathematics, lost to Europe during the Dark Ages, on their journey to India and back through Moorish channels to Southern Europe, having acquired an algebraical tendency on their way.

Last term a boy in the Classical V asked me, "Isn't every parallelogram a rectangle?" Instead of merely answering "No!" I pointed out that *rectus* meant *upright* in both the physical and moral sense, an *upright* stick gave us our definition of *right* angle. We traced its connection of form and meaning with the English *right* and the French *droit* (< *directum*). When Milton wrote,

> "Mammon
> The least erected spirit that fell
> From heaven",

he was introducing a word of the same origin with a moral meaning, a word which survives to-day only in the concrete.

Later, when that boy meets the word *orthogonal*, he may be reminded that the Greek ὀρθός also means *upright* in both the physical and moral sense.

Arc and *chord* (the *bow* and the *string*) are more obvious cases of the borrowing of names of concrete object for geometrical ideas. In French phraseology the figure is complete, *flèche* (the *arrow*) is used for the height of the arc.

These examples, which could be multiplied, show how even in the mathematical class-room the pupil may study the development of

the form of words and of their meanings. They show that there is a parallelism in the development of ideas and of language among different nations. And they provide peep-holes in the partition that separates mathematics from literary studies, whereby both mathematics and language studies profit.

Occasionally an example occurs which emphasises the fact that while in literary use the meaning of a word is fluid and changes as ideas progress, in mathematics it must be crystallised and permanent.

Another Fifth Form boy said to me, "How can a triangle be equilateral? It must be equilateral to some other triangle". That was a singular and illuminating question for a boy to ask at that stage of his school life. I explained to him that *equilateral* was restricted to the equality of sides of one triangle, while *equiangular* was left free for the equality of corresponding angles in different triangles. In somewhat the same way the word *hypotenuse*, the Greek participle ὑποτεινούσῃ, of which *subtending* is an exact translation and equivalent, is always used by Euclid where we should say *subtending*. Using the Latin word for general application, we restrict the Greek word to the case of the right angle.

On the face of it, *parallelogram* means a figure bounded by pairs of parallel sides; but its use is restricted to the quadrilateral.

Contrast this with the range of meaning covered by words of the same origin: a *candid* friend, *candidate*, *incandescent*, *candle*; or with words like *egregious*, *examine*, *zest*, whose meanings have departed so far from their face value that few people think of their origin as they use them.

And just occasionally a word starts a train of ideas not literary but rather scientific. When the word *projection* is introduced in geometry, I make it a text for a short lecture. Mention is made of the camera which projects the landscape on to a photographic film, of nature's camera the eye which projects it on to the retina, of the cinema lantern which projects a tiny picture on to a large screen, of the artist sketching who projects the landscape on to a transparent and imaginary screen which is situated at the distance of his outstretched arm from his eye and on which by an extended pencil he measures angles and lengths. The lecture also refers to the simpler ways of projecting a piece of the earth's spherical surface on to a flat map.

The history of mathematics offers endless invitations to stray awhile from the syllabus to study the development of the ideas and symbolism of the subject. Names crop up: Pythagoras, Apollonius and others. Are they to be merely names to our pupils, or shall we clothe the shadowy names with what substance we can? If we do, we shall not omit to tell those anecdotes which invest the substance with a human interest: the story of Thales' donkey, of the bolting of Pascal's horse, of de Moivre's death from following the terms of an A.P.

We may invite our class to visit in imagination the school of Pythagoras and discuss with him and his pupils the geometry of a

pavement formed by (i) plain square tiles, (ii) square tiles with a diagonal pattern. Starting with the assumption that the tile has four right angles and that its sides are equal, we "discover" for special cases, *e.g.* the isosceles right-angled triangle, most of the elementary geometry as far as Pythagoras' Theorem. By considering other diagonal lines, we generalise these facts to cover any right-angled triangle and any parallelogram. We carry the generalisation farther by considering any triangle as being made up of two right-angled triangles. We may now with a suitable class or at a suitable stage discuss the question of a suitable order for the theorems; we can show by means of the congruence of triangles and the symmetry of the isosceles triangle that our original assumption might have been narrowed down to the existence of an equilateral quadrilateral having two adjacent angles right angles. This involves one more than the five data necessary for the determination of a quadrilateral, the extra datum being equivalent to the assumption of the parallel-axiom.

I like sometimes to devote a lesson to giving a batch of proofs of Pythagoras' Theorem.

I like to devote one to some account of the story of π.

At another time the class are asked to consider the priests of a burgeoning civilisation, making use of the opportunities of their caste to observe through the day and from day to day the shadows cast by a pillar (a gnomon). Acquiring thus some knowledge of the motion of the sun and the length of the year they orient their temples so as to determine the arrival of the seasons. Here the class see the priests as the first scientists and as the repository of scientific knowledge in such forms as the Ahmes papyrus.

The congruence of the base-line and two adjacent angles will lead to some talk of surveying; of rangefinding; of nature's rangefinder possessed by every two-eyed being, the eyes being observation posts and theodolites, the interpupillar distance being the base-line. As the surveyor requires tables for solving his problems, and the range-finder has the necessary tables engraved on it, so the living range-finder requires the tables that the brain has collected from a long series of infantile experiments and engraved on the memory. It will be significant that while the rangefinder has existed in man from the Creation, one of the first contributions to theoretical geometry was Thales' statement of the corresponding theorem.

There is neither time nor need for me to enlarge on this part of my subject. There is a considerable literature available: Rouse Ball, Cajori, Lodge, Hart, Sullivan, D. E. Smith and others all supply material suitable for the elementary class-room. A book by Dr. Vera Sanford, recently published, seems to me to offer in the most ample and convenient form what the teacher will find useful to refer to or a pupil attractive to read.

At times we are tempted to wander into generalisations: questions of congruence or construction lead to the investigation of the number of data necessary to determine a polygon or polyhedron of n vertices; the identity $(a+b)^2 \equiv a^2+2ab+b^2$ leads to

$$(a+b+c+d+\ldots)^2 \equiv a^2+b^2+c^2+d^2+\ldots$$
$$+2ab+2ac+2ad+\ldots$$
$$+2bc+2bd+\ldots$$
$$+2cd+\ldots.$$

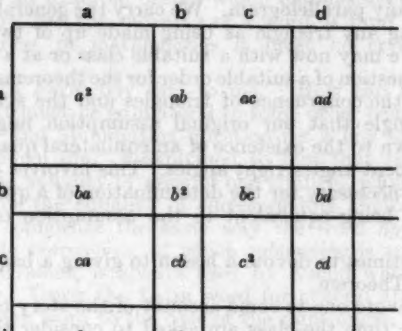


FIG. 2.

Here the columns in the algebraical expression correspond to the gnomons in the geometrical figure, and the algebraical arrangement of the terms provides an orderly and simple way of writing down the expression in its entirety for any number of letters.

In another direction the identity leads through $(a+b)^3$, $(a+b)^4$, etc., to Pascal's Triangle and the Binomial Theorem. Pascal's Triangle is worth a lesson in itself, containing as it does all values of ${}^nC_r$, the binomial expansions for all integral indices positive or negative and the summation of certain series.

To extend the identity $a^2-b^2=(a+b)(a-b)$ by the divisions $\frac{a^3-b^3}{a-b}$, $\frac{a^4-b^4}{a-b}$, etc., is to give it a different background; incidentally it leads to the summation of a G.P. and to finding $\frac{dx^n}{dx}$ where n is a positive integer.

Again, you may have a class that will appreciate those neat proofs that do not appear in elementary text-books but are elementary in their nature: that the number of primes is infinite; that there are only three regular figures such that a number of one of them will fill the plane space about a point; that there are only five regular solids; that a polygon of given perimeter has not its maximum area if two sides are unequal; that there is no fraction with integral numerator and denominator whose value $=\sqrt{2}$; that for any polyhedron

$$E+F=V+2.$$

Finally, or nearly finally, I should like to say something about Puzzles. The boundary between a puzzle and a problem I cannot determine. What is a problem to Smith who knows some mathematics may be a puzzle to Jones who knows rather less. But I should not regard the problem as a puzzle unless it was sufficiently seductive to tempt Jones to tackle it. It must be more exciting than the text-book problem. I rather suspect that the problems of the Ahmes papyrus were used as posers for the presumptuous layman and " whetstones of wit " for the acolytes of the temple. Dr. Vera Sanford's history has an interesting chapter on problems in early mathematical times and goes on to deal with Puzzles, the *problèmes plaisans et délectables* of Bachet. Rouse Ball's *Mathematical Recreations* and Dudeney's collections are probably known to everyone here. The daily papers and monthly magazines propound puzzles to the business man whose mathematical equipment is now, and probably always was, quite inadequate to deal with them. But they tempt him to try them, and he exhibits a persistence he never exhibited in the mathematical class-room, probably because he may now apply wangles that were not legitimate when he was a schoolboy. The rules of the game are now of his own devising.

Some years ago I used to spend an evening each week at a Home for orphans aged fourteen to eighteen. Coming in from their work, they bathed and then read or played quiet games till the time of their evening meal. One evening I found the draughtboard and the billiard table deserted, the latest adventures of Sexton Blake laid aside. With one consent they were trying to solve the following problem : *A man brings home a bag of nuts to divide among his four sons. He goes out and in his absence the eldest son thinks he will take his nuts. He finds there are* $4n+1$ *nuts. He gives the odd nut to the parrot, takes a quarter of the remainder and goes. The second son has the same idea and the same experience. The third son and the fourth do the same thing. Finally the father comes home and counts the nuts. There are* $4m+1$. *What is the smallest number there could have been originally ?* Those boys spent several evenings on a problem they were unlikely to solve except by accident, but they thought they were always approaching a solution and persisted.

I have told you this story because it appears to provide a better clue to the attraction of the puzzle than a case quoted from the class-room.

In a class-room of specialists this puzzle might be reduced to a routine question in Indeterminate Equations, all the excitement wrung out of it, the solution depending on integral solutions of the equation $81x-781=1024y$.

Presented to a class of bright Third-Formers by a colleague—I will call him Mr. Johnson—who makes puzzles a standing dish in the bill of fare for his classes, it has several times been solved thus :

The original number (N) must be one of the sequence

5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45. ... ;

the number remaining after the first division (N_1) being

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, ...

In order that the second division may be possible, the value of N_1 must be one of the sequence N.

Now try for N	13, 29, 45, 61, 77, 93, 109, 125,
getting for N_1	9, 21, 33, 45, 57, 69, 81, 93,
and for N_2	33, 69,

omitting from the N_2 sequence derivates from values of N_1 which we have already seen to be out of the question. The solution is quickly reached on these lines; but, of course, the behaviour of the differences is by now making suggestions for shortening the end-game.

When such a solution is produced and put before a class, it leads up to the following: Let N be $4p+1$ and write it as $4(p+1)-3$, then at the first division the eldest son takes p and gives 1 to the parrot, leaving $3(p+1)-3$; i.e. if N is of the form $M-3$, N_1 is $\frac{3}{4}M-3$, N_2 is $\frac{9}{16}M-3$, N_3 is $\frac{27}{64}M-3$ and N_4 is $\frac{81}{256}M-3$. And since N_4-1 is to be a multiple of 4, M must be a multiple of 1024.

The same solution can be put in this way: Suppose that 3 marbles were put in with nuts so as to make a total count divisible by 4. Then each boy, taking the number of nuts he thinks he is entitled to, giving 1 to the parrot, and leaving the marbles in with the remainder, leaves a count divisible by 4. This happens at 5 stages of division. Therefore including the marbles there must be a multiple of 4^5 or 1024. The number of nuts is $1024n-3$.

The same Mr. Johnson occasionally sets this question: *Miss Smith, a typist, having a date for the evening, is in a state of excitement and hurry. In consequence the five letters she has to post are all put into wrong envelopes. In how many ways could this happen?*

This, which is a not too easy question in permutations for the specialist, is solved by the Third-Former by writing down all the arrangements of the digits 1, 2, 3, 4, 5 and crossing out every arrangement in which any digit occurs in its right order.

A number of conundrums of this type go the rounds: "*Find the code for which this is an exact addition sum*", and from time to time I find myself bombarded from various quarters by the same conundrum. I was even asked if I could do this by a Frenchman in a small village in Savoy.

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 SEND
 MORE
MONEY
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The missing digit problem belongs to the same family and is a great favourite with boys. Rouse Ball has some classic examples, and Berwick's article in the *Gazette* for 1920-1, p. 359, will introduce you to the higher considerations connected with them. Boys like composing them and exhibit a sense of humour as well as ingenuity in their compositions.

There are a number of digital problems, of which perhaps the most exciting is that of the four 4's: *To make combination of four 4's using any recognised signs, including the* $\surd$ *and* !, *to represent any integer from* 1 *to* 100.

Boys will keep their solutions to this problem, and return to them with improvements and additions over a long period. Other boys have a taste for Magic Squares and seem to find an inexhaustible interest in them.

Is it not possible that when our ordinary baits fail to hook our fish, a puzzle may be the right fly to cast? Certainly boys will call up unsuspected resources and interests to solve some puzzle that happens to appeal to them.

The Third-Former's solution of the nut question gives evidence of the existence and exercise of a faculty which would find little chance of expression in routine class-work, but which would be invaluable in a man who had to show initiative in tackling a novel problem. And I doubt whether any text-book problem would incite an average Third-Former to the sustained effort he will make to solve a puzzle.

You are probably saying by now that you agree that the mathematical lesson will be more educative and more exciting if you can spare time for these wanderings, but where is the time to come from? I can tell you that my Mr. Johnson wastes more time in this way than anyone I know and gets more work on the syllabus out of his boys. The secret appears to lie in this, that they go to his room with an expectation of something interesting and pleasurable. Their attention is more strongly enlisted and therefore more productive. And here is an interesting sidelight on his methods: if a backward boy wants cramming for an examination, he asks if Mr. Johnson can do the cramming.

And here is my real "finally". What of the sidetrack that a boy tries to start you on and that leads, as far as you can see, to no educational goal? I remember very vividly a class I was in as a boy where the boy who couldn't understand was caned till he did—or pretended to. Gentler methods are used to-day. But there are boys that defeat our efforts. They are carried from class to class by the pressure of boys entering the school from term to term as a pebble is carried by a mountain stream. They develop into listless and hopeless beings, passing (as R. L. Stevenson says) "lack lustre periods between sleeping and waking in the class". If I can wake such fellows into alertness or interest by an occasional digression, I am willing to do it. But I must say they show little disposition to waste my time and that of the class on aimless suggestions. And on the whole they respond surprisingly well to the digressions on which I like to lead them. I should say that even in the B Class, where one has apparently less time to spare for them, the sidetracks pay.

F. C. B.

GLEANINGS FAR AND NEAR.

898. Several competitors have asked the meaning of the symbol π (pronounced pie) and the reason why 4π is equal to 12·57.

π is a natural number. If we divide the circumference of any circle by its diameter we shall always obtain the same number 3·1416. This number is indicated for short as π.—*World-Radio*, May 13, 1932, p. 951.

AN EXTENSION OF VAN DER MOND'S THEOREM AND SOME APPLICATIONS.

By N. R. C. Dockeray.

The following extension of Van der Mond's Theorem enables us to express as an infinite series one of the solutions of the equation

$$x^r + ax^s + b = 0.$$

It is true that the results can all be obtained from Lagrange's Theorem, but the process is interesting in that it enables us to obtain the series by an algebraical method.

The extension is

$$\frac{p+q}{p+q+rn}\binom{p+q+rn}{r} = \sum_{s=0}^{r} \frac{p}{p+\overline{r-s}\,n}\binom{p+\overline{r-s}\,n}{r-s}\frac{q}{q+sn}\binom{q+sn}{s}, \quad (1)$$

where p, q, and n are any numbers whatever, r is a positive integer, and $\binom{m}{s}$ is as usual an abbreviation of

$$\frac{m(m-1)(m-2)\dots(m-s+1)}{s!}.$$

In order to establish this result, assume that, for some particular integral value of r, the following identity holds:

$$\sum_{s=1}^{r}\binom{r-1}{s-1}\frac{(p+sn)_s q(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)} = (p+q+rn-1)_{r-1}, \quad \dots\dots(2)$$

in which m_s is used to denote $m(m-1)(m-2)\dots(m-s+1)$. Interchanging p and q, and replacing s by t, we have

$$\sum_{t=1}^{r}\binom{r-1}{t-1}\frac{(q+tn)_t p(p+\overline{r-t}\,n)_{r-t}}{(q+tn)(p+\overline{r-t}\,n)} = (p+q+rn-1)_{r-1}.$$

Now write $t = r-s$, and use $\binom{r-1}{r-s-1} = \binom{r-1}{s}$; we have

$$\sum_{s=0}^{r-1}\binom{r-1}{s}\frac{p(p+sn)_s(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)} = (p+q+rn-1)_{r-1}, \quad \dots\dots(3)$$

Now multiply equation (2) by $p+rn-r$ and equation (3) by q, and add the results, we get

$$\binom{r-1}{0}\frac{q(q+rn)_r}{q+rn} + \sum_{s=1}^{r-1}\left\{\frac{(p+sn)_s(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)}\left[\binom{r-1}{s-1}q(p+rn-r)+\binom{r-1}{s}pq\right]\right\} + \binom{r-1}{r-1}\frac{(p+rn-r)(p+rn)_r q}{(p+rn)q} = (p+q+rn-1)_r.$$

Now $$\binom{r-1}{s-1}(p+rn-r) = \binom{r-1}{s-1}p + \binom{r}{s}s(n-1).$$

Hence

$$\binom{r-1}{0}\frac{q(q+rn)_r}{q+rn}+\sum_{s=1}^{r-1}\left\{\binom{r-1}{s-1}+\binom{r-1}{s}\right\}\frac{p(p+sn)_s q(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)}$$
$$+\sum_{s=1}^{r-1}\binom{r}{s}\frac{(ns-s)(p+sn)_s q(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)}+\binom{r-1}{r-1}\frac{(p+rn)_{r+1}}{p+rn}$$
$$=(p+q+rn-1)_r,$$

i.e.
$$\binom{r}{0}\frac{q(q+rn)_r}{q+rn}+\sum_{s=1}^{r-1}\binom{r}{s}\frac{(p+sn-s)(p+sn)_s q(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)}$$
$$+\binom{r}{r}\frac{(p+rn)_{r+1}}{p+rn}=(p+q+rn-1)_r,$$

i.e.
$$\sum_{s=0}^{r}\binom{r}{s}\frac{(p+sn)_{s+1}q(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)}=(p+q+rn-1)_r.$$

In this result, which, according to our assumption, is true for all values of p, write $p+n$ for p and $s-1$ for s, we thus get

$$\sum_{s=1}^{r+1}\binom{r}{s-1}\frac{(p+sn)_s q(q+\overline{r+1-s}\,n)_{r+1-s}}{(p+sn)(q+\overline{r+1-s}\,n)}=(p+q+\overline{r+1}\,n-1)_r, \quad ...(4)$$

which is the same as (2), with $r+1$ instead of r.

Now taking the case $r=3$, we have

$$\binom{2}{0}\frac{(p+n)q(q+2n)(q+2n-1)}{(p+n)(q+2n)}+\binom{2}{1}\frac{(p+2n)(p+2n-1)q(q+n)}{(p+2n)(q+n)}$$
$$+\binom{2}{2}\frac{(p+3n)(p+3n-1)(p+3n-2)q}{(p+3n)q}$$
$$=q(q+2n-1)+2(p+2n-1)q+(p+3n-1)(p+3n-2)$$
$$=(p^2+2pq+q^2)+(6n-3)(p+q)+(3n-1)(3n-2)$$
$$=(p+q+3n-1)(p+q+3n-2),$$

so that equation (2) holds when $r=3$. It follows by the usual argument of mathematical induction that it is true for all positive integral values of r.

If we now multiply equation (2) by p and equation (3) by q, and add the results, we get

$$\binom{r-1}{0}\frac{pq(q+rn)_r}{p(q+rn)}+\sum_{s=1}^{r-1}\left\{\binom{r-1}{s-1}+\binom{r-1}{s}\right\}\frac{p(p+sn)_s q(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)}$$
$$+\binom{r-1}{r-1}\frac{p(p+rn)_r q}{(p+rn)q}=(p+q)(p+q+rn-1)_{r-1},$$

i.e.
$$\sum_{s=0}^{r}\binom{r}{s}\frac{p(p+sn)_s q(q+\overline{r-s}\,n)_{r-s}}{(p+sn)(q+\overline{r-s}\,n)}=\frac{p+q}{p+q+rn}(p+q+rn)_r.$$

Dividing all through by $r!$ we have

$$\sum_{s=0}^{r}\frac{p}{p+sn}\binom{p+sn}{s}\frac{q}{q+\overline{r-s}\,n}\binom{q+\overline{r-s}\,n}{r-s}=\frac{p+q}{p+q+rn}\binom{p+q+rn}{r},$$

which is the required result.

Another identity of a similar nature can be obtained from equation (4). In this write $r+1-s$ for s, and we get

$$\sum_{s=0}^{r}\binom{r}{r-s}\frac{(p+\overline{r+1-s}\,n)_{r+1-s}q(q+sn)_s}{(p+\overline{r+1-s}\,n)(q+sn)}=(p+q+\overline{r+1}\,n-1)_r.$$

Now put $p=(t-1)n+1$; we obtain

$$\sum_{s=0}^{r}\binom{r}{s}\frac{(\overline{t+r-s}\,n+1)_{r+1-s}q(q+sn)_s}{(\overline{t+r-s}\,s+1)(q+sn)}=(q+\overline{t+r}\,n)_r,$$

i.e.

$$\sum_{s=0}^{r}\binom{\overline{m-s}\,n}{r-s}\frac{q}{q+sn}\binom{q+sn}{s}=\binom{q+mn}{r},$$

where m is written for $t+r$.

Thus if s is a positive integer, and p, r, n any numbers,

$$\binom{p+rn}{s}=\sum_{k=0}^{s}\binom{\overline{r-k}\,n}{s-k}\frac{p}{p+kn}\binom{p+kn}{k}. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Consider now the series

$$\sum_{r=0}^{\infty}\frac{p}{p+rn}\binom{p+rn}{r}x^r=1+px+\frac{p(p+2n-1)}{2!}x^2 + \frac{p(p+3n-1)(p+3n-2)}{3!}x^3+\ldots. \quad \ldots(6)$$

If n is a rational number it is not difficult to determine the radius of convergence of this series. For suppose $n=\frac{\lambda}{\mu}$, where λ and μ are integers, and μ is positive. If the series be denoted by

$$u_0+u_1+u_2+\ldots+u_r+\ldots,$$

then

$$u_r=\frac{p}{p+rn}\binom{p+rn}{r}x^r = \frac{p\left(p+\frac{\lambda}{\mu}r-1\right)\left(p+\frac{\lambda}{\mu}r-2\right)\ldots\left(p+\frac{\lambda}{\mu}r-r+1\right)}{r!}x^r,$$

and

$$u_{r+\mu}=\frac{p}{p+\overline{r+\mu}\,n}\binom{p+\overline{r+\mu}\,n}{r+\mu}x^{r+\mu} = \frac{p\left(p+\frac{\lambda}{\mu}r+\lambda-1\right)\left(p+\frac{\lambda}{\mu}r+\lambda-2\right)\ldots\left(p+\frac{\lambda}{\mu}r+\lambda-\mu-r+1\right)}{(r+\mu)!}x^{r+\mu}.$$

Hence

$$\frac{u_r}{u_{r+\mu}}=\frac{(p+nr-1)(p+nr-2)\ldots(p+nr-r+1)(r+1)(r+2)\ldots(r+\mu)}{(p+nr+\lambda-1)(p+nr+\lambda-2)\ldots(p+nr+\lambda-\mu-r+1)\,x^{\mu}}.$$

Various cases now arise.

(*a*) Suppose λ positive and $> \mu$, then

$$\frac{\frac{u_r}{u_{r+\mu}}}{} = \frac{(p+nr+\lambda-\mu-r)(p+nr+\lambda-\mu-r-1)\dots(p+nr-r+1)(r+1)(r+2)\dots(r+\mu)}{(p+nr+\lambda-1)(p+nr+\lambda-2)\dots(p+nr)\,x^\mu},$$

the remaining factors having cancelled out.

From this we find

$$\operatorname*{Lim}_{r\to\infty}\frac{u_r}{u_{r+\mu}} = \frac{(n-1)^{\lambda-\mu}}{n^\lambda x^\mu} = \left(\frac{(n-1)^{n-1}}{n^n x}\right)^\mu.$$

Hence if $|x| < \dfrac{(n-1)^{n-1}}{n^n}$, each of the series

$$u_0 + u_\mu + u_{2\mu} + u_{3\mu} + \dots,$$
$$u_1 + u_{\mu+1} + u_{2\mu+1} + u_{3\mu+1} + \dots,$$
$$u_2 + u_{\mu+2} + u_{2\mu+2} + u_{3\mu+2} + \dots, \text{ etc.},$$

is absolutely convergent, hence the whole series is absolutely convergent.

(*b*) Suppose λ positive and $< \mu$, then we find as above that the series is convergent if $|x| < \dfrac{1}{n^n(1-n)^{1-n}}$.

(*c*) Suppose λ negative, we find that the series is convergent if

$$|x| < \frac{(-n)^{(-n)}}{(1-n)^{1-n}}.$$

Hence the radius of convergence of the series

$$1 + px + \frac{p}{p+2n}\binom{p+2n}{2}x^2 + \frac{p}{p+3n}\binom{p+3n}{3}x^3 + \dots$$

is $\dfrac{(n-1)^{n-1}}{n^n}$, if $n > 1$,

$\dfrac{1}{n^n(1-n)^{1-n}}$, if $0 < n < 1$,

and $\dfrac{m^m}{(m+1)^{m+1}}$, if n is negative, and equal to $-m$.

If $n = 0$ or 1, the radius of convergence is unity.

Now suppose x to have a value such that the series is convergent, then, if we write

$$F(p) = \sum_{s=0}^{\infty}\frac{p}{p+sn}\binom{p+sn}{s}x^s,$$

we have $F(p) \times F(q) = F(p+q)$, by equation (1).

Hence by the argument used in the proof of the Binomial and Exponential Series, we have

$$F(p) = \{F(1)\}^p,$$

provided that p is rational.

Hence if

$$y = F(1) = 1 + x + \frac{nx^2}{1!} + \frac{n(3n-1)}{2!}x^3 + \frac{n(4n-1)(4n-2)}{3!}x^4 + \dots,$$

we have

$$y^p = 1 + px + \frac{p(p+2n-1)}{2!}x^2 + \frac{p(p+3n-1)(p+3n-2)}{3!}x^3 + \dots,$$

so that

$$y^n = 1 + nx + \frac{n(3n-1)}{2!}x^2 + \frac{n(4n-1)(4n-2)}{3!}x^3 + \dots$$

$$= \frac{y-1}{x}.$$

Thus $$y = 1 + xy^n.$$

Hence within its circle of convergence the series for y represents that root of the equation for y which tends to 1 as $x \to 0$.

This result can be transformed in various ways. Thus if

$$y = a + xy^n$$

where a is positive, we have, writing $y = au$,

$$u = 1 + (a^{n-1}x)u^n,$$

so that $$u^m = \sum_{r=0}^{\infty} \frac{m}{m+rn}\binom{m+rn}{r}(a^{n-1}x)^r,$$

i.e. $$y^m = \sum_{r=0}^{\infty} \frac{m}{m+rn}\binom{m+rn}{r}a^{m+rn-r}x^r.$$

Again, if $y = -a + xy^n$, where a and x are positive, we have

$$xy^n = a + y.$$

Now let $$y = \left(\frac{au}{x}\right)^{\frac{1}{n}};$$

then $$x\frac{au}{x} = a + \frac{a^{\frac{1}{n}}}{x^{\frac{1}{n}}}u^{\frac{1}{n}},$$

i.e. $$u = 1 + \frac{a^{\frac{1-n}{n}}}{x^{\frac{1}{n}}}u^{\frac{1}{n}},$$

and from this we obtain an expansion in powers of $x^{-\frac{1}{n}}$.

Or, if an expansion in positive powers of x be required, we have, putting $y = (vx)^{\frac{1}{1-n}}$,

$$v^{-\frac{1}{n-1}}x^{-\frac{1}{n-1}} = -a + xv^{-\frac{n}{n-1}}x^{-\frac{n}{n-1}}$$

$$= -a + x^{-\frac{1}{n-1}}v^{-\frac{n}{n-1}},$$

so that, multiplying by $v^{\frac{n}{n-1}}x^{\frac{1}{n-1}}$,

$$v = 1 - ax^{\frac{1}{n-1}}v^{\frac{n}{n-1}},$$

whence an expansion for v^m can be obtained in powers of $-ax^{\frac{1}{n-1}}$.

In the same way, if $y^n + ay = x$, where a and x are positive, we have, putting $y = \frac{xu}{a}$,

$$\frac{x^n u^n}{a^n} + xu = x,$$

i.e.
$$u = 1 - a^{-n}x^{n-1}u^n,$$

whence the required expansion is obtained.

As an example, consider the equation $z^5 + az^3 = x$.

Let
$$z^3 = \frac{xu}{a};$$

whence
$$\left(\frac{xu}{a}\right)^{\frac{5}{3}} + xu = x$$

i.e.
$$u = 1 - \frac{x^{\frac{2}{3}}}{a^{\frac{5}{3}}}u^{\frac{5}{3}}.$$

Hence

$$u^m = 1 + \frac{m}{m+\frac{5}{3}}\binom{m+\frac{5}{3}}{1}\left(-\frac{x^{\frac{2}{3}}}{a^{\frac{5}{3}}}\right) + \frac{m}{m+\frac{10}{3}}\binom{m+\frac{10}{3}}{2}\left(-\frac{x^{\frac{2}{3}}}{a^{\frac{5}{3}}}\right)^2 + \frac{m}{m+\frac{15}{3}}\binom{m+\frac{15}{3}}{3}\left(-\frac{x^{\frac{2}{3}}}{a^{\frac{5}{3}}}\right)^3 + \dots$$

$$= 1 - \frac{3m}{3}\left(\frac{x^2}{a^5}\right)^{\frac{1}{3}} + \frac{3m(3m+7)}{3.6}\left(\frac{x^2}{a^5}\right)^{\frac{2}{3}} - \frac{3m(3m+12)(3m+9)}{3.6.9}\left(\frac{x^2}{a^5}\right)^{\frac{3}{3}} + \frac{3m(3m+17)(3m+14)(3m+11)}{3.6.9.12}\left(\frac{x^2}{a^5}\right)^{\frac{4}{3}} - \dots.$$

In particular $z = \frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}u^{\frac{1}{3}}$

$$= \frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}} - \frac{1}{3}\frac{x^{\frac{3}{3}}}{a^{\frac{6}{3}}} + \frac{1.8}{3.6}\frac{x^{\frac{5}{3}}}{a^{\frac{11}{3}}} - \frac{1.13.10}{3.6.9}\frac{x^{\frac{7}{3}}}{a^{\frac{16}{3}}} + \frac{1.18.15.12}{3.6.9.12}\frac{x^{\frac{9}{3}}}{a^{\frac{21}{3}}} - \dots \dots \quad (7)$$

It can easily be verified, by putting $m = \frac{5}{3}$ and $m = 1$ in the series for u^m, that the equation $z^5 + az^3 = x$ is satisfied.

Since n in this case is $\frac{5}{3}$, the series for the various powers of z are convergent provided

$$\left|\frac{x^{\frac{2}{3}}}{a^{\frac{5}{3}}}\right| < \frac{(\frac{2}{3})^{\frac{2}{3}}}{(\frac{5}{3})^{\frac{5}{3}}},$$

i.e.
$$\left|\frac{x^2}{a^5}\right| < \frac{2^2.3^3}{5^5},$$

i.e.
$$|x| < \frac{6\sqrt{3}}{25\sqrt{5}}a^{\frac{5}{2}}.$$

Thus this particular series is valid only for small values of x, and is therefore unsuitable for solving (for example) the equation

$$z^5+z^3=1.$$

A series valid for large values of x may be obtained as follows.

Put $$z=(xv)^{\frac{1}{3}};$$

then $$xv+a(xv)^{\frac{2}{3}}=x,$$

i.e. $$v+ax^{-\frac{2}{3}}v^{\frac{2}{3}}=1,$$

i.e. $$v=1+\left(-\frac{a}{x^{\frac{2}{3}}}\right)v^{\frac{2}{3}}.$$

From this we find

$$v^m=1-\frac{5m}{5}\frac{a}{x^{\frac{2}{3}}}+\frac{5m(5m+1)}{5.10}\frac{a^2}{x^{\frac{4}{3}}}-\frac{5m(5m+4)(5m-1)}{5.10.15}\frac{a^3}{x^2}$$
$$+\frac{5m(5m+7)(5m+2)(5m-3)}{5.10.15.20}\frac{a^4}{x^{\frac{8}{3}}}.$$

This series is convergent provided

$$\left|\frac{a}{x^{\frac{2}{3}}}\right|<\frac{1}{(\frac{2}{5})^{\frac{2}{3}}(\frac{3}{5})^{\frac{3}{3}}},$$

i.e. $$\left|\frac{a^5}{x^2}\right|<\frac{5^5}{3^3.2^2},$$

i.e. $$x>\frac{6\sqrt{3}}{25\sqrt{5}}a^{\frac{5}{2}}.$$

We thus obtain

$$z=t-\frac{1}{5}\frac{a}{t}+\frac{1.2}{5.10}\frac{a^2}{t^3}-\frac{1.5.0}{5.10.15}\frac{a^3}{t^5}+\frac{1.8.3.(-2)}{5.10.15.20}\frac{a^4}{t^7}-\ldots,\ldots(8)$$

where $t=x^{\frac{1}{3}}$.

If $a=x=1$, the first six terms give

$$z=0{\cdot}837504,$$

which is rather too small, giving

$$z^5+z^3=0{\cdot}99957.$$

The series (7) gives the three values of z which tend to zero as $x\to 0$. If the values which tend to $\sqrt{-a}$ are required, we proceed as follows.

Since $$z^5+az^3=x;$$

we have $$z^2+a=xz^{-3}.$$

Put $z=a^{\frac{1}{2}}it^{\frac{1}{3}}$, where $i=\sqrt{-1}$; then

$$t=1-\frac{ix}{a^4}t^{-\frac{1}{3}},$$

whence

$$t^m = 1 - \frac{2m}{2}\frac{ix}{a^4} + \frac{2m(2m-8)}{2.4}\left(\frac{ix}{a^4}\right)^2 - \frac{2m(2m-11)(2m-13)}{2.4.6}\left(\frac{ix}{a^4}\right)^3 + \frac{2m(2m-14)(2m-16)(2m-18)}{2.4.6.8}\left(\frac{ix}{a^4}\right)^4 - \dots,$$

so that

$$z = ia^{\frac{1}{2}} + \frac{1}{2}\frac{x}{a^{\frac{7}{2}}} + \frac{1.7}{2.4}\frac{ix^2}{a^{\frac{13}{2}}} - \frac{1.10.12}{2.4.6}\frac{x^3}{a^{\frac{19}{2}}} - \frac{1.13.15.17}{2.4.6.8}\frac{ix^4}{a^{\frac{25}{2}}} + \dots.$$

This series converges if $|x| < \frac{6\sqrt{3}}{25\sqrt{5}}a^4$.

The method can also be used to obtain in a very general form the expansions of circular functions of multiple angles.

Let $$y = \sqrt{1+x^2} + x.$$

Then $$y^2 - 2xy - 1 = 0.$$

Hence if $y = u^{\frac{1}{2}}$, we have

$$u = 1 + 2x.u^{\frac{1}{2}},$$

so that

$$u^{m/2} = 1 + \frac{m/2}{m/2+\frac{1}{2}}\binom{m/2+\frac{1}{2}}{1}2x + \frac{m/2}{m/2+1}\binom{m/2+1}{2}4x^2 + \frac{m/2}{m/2+\frac{3}{2}}\binom{m/2+\frac{3}{2}}{3}8x^3 + \dots.$$

$$= 1 + \frac{m}{1}x + \frac{m.m}{1.2}x^2 + \frac{m(m+1)(m-1)}{1.2.3}x^3 + \frac{m(m+2).m(m-2)}{1.2.3.4}x^4 + \dots.$$

Hence

$$y^m = 1 + \frac{m}{1}x + \frac{m^2}{2!}x^2 + \frac{m(m^2-1^2)}{3!}x^3 + \frac{m^2(m^2-2^2)}{4!}x^4 + \dots. \quad \dots\dots(9)$$

This expansion is valid provided

$$|2x| < \frac{1}{(\frac{1}{2})^{\frac{1}{2}}(\frac{1}{2})^{\frac{1}{2}}},$$

i.e. $$|x| < 1.$$

We are therefore at liberty to put $x = i\sin\theta$, then

$$y = \cos\theta + i\sin\theta,$$

hence

$$\cos m\theta + i\sin m\theta = 1 + \frac{m}{1!}i\sin\theta - \frac{m^2}{2!}\sin^2\theta - \frac{m(m^2-1^2)}{3!}i\sin^3\theta + \frac{m^2(m^2-2^2)}{4!}\sin^4\theta + \frac{m(m^2-1^2)(m^2-3^2)}{5!}i\sin^5\theta - \dots,$$

whence we obtain the well-known results :

$$\left.\begin{aligned}\cos m\theta &= 1 - \frac{m^2}{2!}\sin^2\theta \\ &\quad + \frac{m^2(m^2-2^2)}{4!}\sin^4\theta - \frac{m^2(m^2-2^2)(m^2-4^2)}{6!}\sin^6\theta + \dots \\ \sin m\theta &= m\sin\theta - \frac{m(m^2-1^2)}{3!}\sin^3\theta \\ &\quad + \frac{m(m^2-1^2)(m^2-3^2)}{5!}\sin^5\theta - \dots\dots\dots\dots\end{aligned}\right\} \dots(10)$$

This proof of course establishes these series only for real rational values of m ; for irrational or complex values, recourse must be had to other methods.

Consider now the product of the two series :

$$1 + \frac{m}{m+n}\binom{m+n}{1}x + \frac{m}{m+2n}\binom{m+2n}{2}x^2 + \frac{m}{m+3n}\binom{m+3n}{3}x^3 + \dots$$

and $$\binom{kn}{0} + \binom{\overline{k+1}\,n}{1}x + \binom{\overline{k+2}\,n}{2}x^2 + \binom{\overline{k+3}\,n}{3}x^3 + \dots \dots\dots\dots(11)$$

Let $$y = 1 + xy^n,$$

then the first series is y^m : let the second be denoted by $f(k, n ; x)$. The coefficient of x^s in $y^m f(k, n ; x)$ is

$$\binom{kn}{0}\frac{m}{m+sn}\binom{m+sn}{s} + \binom{\overline{k+1}\,n}{1}\frac{m}{m+\overline{s-1}\,n}\binom{m+\overline{s-1}\,n}{1} + \dots$$
$$+ \binom{\overline{k+s-1}\,n}{s-1}\frac{m}{m+n}\binom{m+n}{1} + \binom{\overline{k+s}\,n}{s},$$

i.e. $\binom{m+\overline{k+s}\,n}{s}$ by equation (5).

Put $m = \lambda n$, the coefficient of x^s is then

$$\binom{\overline{\lambda+k+s}\,n}{s},$$

hence we have

$$y^{\lambda n} f(k, n ; x) = f(k+\lambda, n ; x).$$

Thus $$f(\lambda, n ; x) = y^{\lambda n} f(0, n ; x).$$

Now $$f(0, n ; x) = 1 + \binom{n}{1}x + \binom{2n}{2}x^2 + \binom{3n}{3}x^3 + \dots,$$

and $$f\left(\frac{n-1}{n}, n ; x\right) = 1 + \binom{2n-1}{1}x + \binom{3n-1}{2}x^2 + \binom{4n-1}{3}x^3 + \dots ;$$

hence $$nxf\left(\frac{n-1}{n}n ; x\right) = \binom{n}{1}x + \binom{2n}{2}x^2 + \binom{3n}{3}x^3 + \dots$$
$$= f(0, n ; x) - 1.$$

Thus $$f(0, n ; x) - nxf\binom{n-1}{n}, n ; x) = 1.$$

But $$f\left(\frac{n-1}{n}, n; x\right) = y^{n-1} f(0, n; x),$$

therefore $$f(0, n; x) = \frac{1}{1 - nxy^{n-1}}$$

and $$f(p, n; x) = \frac{y^{pn}}{1 - nxy^{n-1}}.$$

Now put $n = \frac{1}{2}$, and write $2x$ for x; we have then

$$f(p, \tfrac{1}{2}; 2x) = \frac{u^{2p}}{1 - xu^{-1}},$$

where $$u = 1 + 2xu^{\frac{1}{2}}.$$

Hence if $$y = \sqrt{1+x^2} + x, \quad = u^{\frac{1}{2}},$$

$$f(p-1, \tfrac{1}{2}; 2x) = \frac{y^{2p-1}}{1 - xy^{-1}} = \frac{y^{2p}}{y - x}$$

$$= \frac{(\sqrt{1+x^2}+x)^{2p}}{\sqrt{1+x^2}}.$$

We thus have

$$\frac{(\sqrt{1+x^2}+x)^{2p}}{\sqrt{1+x^2}} = \binom{\frac{1}{2}p-1}{0} + \binom{\frac{1}{2}p}{1} 2x + \binom{\frac{1}{2}p+1}{2} (2x)^2 + \binom{\frac{1}{2}p+2}{3} (2x)^3 + \ldots$$

$$= 1 + \frac{p}{1!}x + \frac{p^2-1^2}{2!}x^2 + \frac{p(p^2-2^2)}{3!}x^3 + \frac{(p^2-1^2)(p^2-3^2)}{4!}x^4 + \ldots . \quad (12)$$

In this put $x = i \sin\theta$, we thus get

$$\left.\begin{aligned} \frac{\cos p\theta}{\cos\theta} &= 1 - \frac{p^2-1^2}{2!}\sin^2\theta + \frac{(p^2-1^2)(p^2-3^2)}{4!}\sin^4\theta - \ldots\ldots\ldots\ldots \\ \frac{\sin p\theta}{\cos\theta} &= p\sin\theta - \frac{p(p^2-2^2)}{3!}\sin^3\theta + \frac{p(p^2-2^2)(p^2-4^2)}{5!}\sin^5\theta - \ldots \end{aligned}\right\} . \quad (13)$$

N. R. C. D.

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899. Mon père traçait à la craie sur un tableau noir des signes énigmatiques, figures de géométrie ou formules d'algèbre, avec cette netteté dans les lignes des courbes ou les lettres des polynômes qui révélait l'habituelle méthode de son être intime. D'autres fois il écrivait, debout, à une table d'architecte qu'il préférait à son bureau,—table composée simplement d'une large planche en bois blanc placée sur deux tréteaux. Les grands livres de mathématiques rangés avec minutie dans la bibliothèque, les figures froides des savants dont les portraits gravés en taille-douce et sous verre étaient les seuls objets d'art dont se décorassent les murs, la pendule qui représentait un globe du monde, deux cartes astronomiques pendues au-dessus du bureau, et, sur ce bureau, la règle à calculs avec ses chiffres et son coulant de cuivre, les équerres, les compas, la règle plate en forme de T, j'évoque à mon gré ces menus détails où tout n'était que pensée, et ces images m'aident à comprendre comment dès ma lointaine enfance le rêve d'une existence purement idéale et contemplative s'élabora en moi.—Paul Bourget, *Le disciple*, ch. iv. [Per Mr. J. R. Bretherton.]

MATHEMATICS AND PSYCHOLOGY.

By Professor H. T. H. Piaggio, M.A., D.Sc.

1. *Introduction.* Psychology has been described as the science which tells us what everyone knows, in language that no one can understand. It has a fatal attraction for those whose paucity of ideas is concealed by an exuberance of vocabulary. The leading authorities not only ignore each others' conclusions, but even refuse to use words in the same sense. "When *I* use a word", said Humpty Dumpty, "it means just what I choose it to mean—neither more nor less." The discussion of such a point as the nature of intelligence, sufficiently difficult in itself, becomes hopeless when everyone attaches a different meaning to that term. It is the object of this article to show how, by the application of mathematical methods, Professor C. Spearman has found a way of escape from this jungle of verbal entanglements. As a preliminary we shall require some account of the idea of *correlation*.

2. *Correlation.* Consider N numbers $A_1, A_2, \ldots$. Denote their arithmetic mean by $\bar{A}$, and the differences $A_1 - \bar{A}, A_2 - \bar{A}, \ldots$ by $a_1, a_2, \ldots$ respectively, so that the arithmetic mean $\bar{a}$ is zero. The *standard deviation* σ_a is defined as $(\Sigma a^2/N)^{\frac{1}{2}}$. If, in addition to $\bar{a}=0$, $\sigma_a=1$, the numbers $a_1, a_2, \ldots$ are said to be expressed in *standard measure*.

Now consider a second set of N numbers $B_1, B_2, \ldots$, and derive $b_1, b_2, \ldots$ and σ_b from them by the methods applied to the A's. The *coefficient of correlation* r_{ab} between the A's and B's is defined as $(\Sigma ab/N\sigma_a\sigma_b)$. It follows from the identity

$$\Sigma\Sigma(a_p b_q - a_q b_p)^2 = (\Sigma a^2)(\Sigma b^2) - (\Sigma ab)^2 = N^2\sigma_a{}^2\sigma_b{}^2(1 - r_{ab}{}^2)$$

that if $r_{ab} = \pm 1$, $a_p/b_p = a_q/b_q$ for all values of p and q, giving a definite linear relation between each A and the corresponding B. As the left-hand side of the identity cannot be negative, r_{ab} cannot lie outside the limits -1 and 1.

In general no such exact linear relation exists, but we may attempt to find an approximate one by choosing λ and μ so that $\Sigma(b - \lambda a - \mu)^2$ is a minimum.

Differentiating partially with respect to λ and μ, this gives

$$-2\Sigma a(b - \lambda a - \mu) = 0 = -2\Sigma(b - \lambda a - \mu),$$

which, as $\Sigma a = N\bar{a} = 0$, and similarly $\Sigma b = 0$, reduce to

$$-2(N\sigma_a\sigma_b r_{ab} - \lambda N\sigma_a{}^2) = 0 = -2(-N\mu)$$

so $\lambda = \sigma_b r_{ab}/\sigma_a$, and $\mu = 0$.

The line
$$b = (\sigma_b r_{ab}/\sigma_a)a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
is called the *line of regression* of B upon A. Interchanging a and b we get the line of regression of A upon B. When $r_{ab} = \pm 1$, these lines coincide, giving the exact linear relation which then connects the A's and B's. When, on the other hand, $r_{ab} = 0$, the regression lines become $b = 0$ and $a = 0$, thus failing to give even an approximate connection. In this case the A's and B's are said to be *uncorrelated*.

If the A's and B's are completely *independent* (*e.g.* the results of throwing two dice), r_{ab} may be expected to approximate to zero if N is large enough. [The converse is not true. A definite non-linear relation like $b^2=a$ can lead to $r_{ab}=0$ if the b's are taken in pairs of equal numerical value and opposite sign, so though *independence* involves *zero correlation* (for sufficiently great N), zero correlation does not necessarily involve independence.]

3. *Tetrad relations.* Now suppose that instead of two sets of N numbers we have K such sets, denoted briefly by $A, B, \dots$, and that these are expressible in terms of $(K+1)$ other sets $G, S_A, S_B, \dots$, no two of which are correlated, and $2K$ constants $m_a, m_b, \dots$; $n_a, n_b, \dots$, by equations such as

$$a = m_a g + n_a s_a, \quad \dots\dots(2)$$

$$b = m_b g + n_b s_b. \quad \dots\dots(3)$$

Each equation really denotes N equations, as a can take any one of the values $a_1, a_2, \dots$, with a corresponding set of values for g and s_a. But m_a and n_a are constants, which occur unaltered in each of the N equations. We shall use the term *averaging* to denote the process of taking the arithmetic mean of the N expressions of a given type. Thus by averaging a, a^2, and ab we get respectively $\bar{a}(=0)$, σ_a^2, and $\sigma_a\sigma_b r_{ab}$. In all that follows we shall assume the numbers reduced to standard measure. These three averages then reduce to 0, 1, and r_{ab} respectively.

From equations (2) and (3) we get

$$ab = m_a m_b g^2 + m_a n_b g s_b + m_b n_a g s_a + n_a n_b s_a s_b,$$

whence, averaging and remembering that g and the s's are mutually uncorrelated,

$$r_{ab} = m_a m_b. \quad \dots\dots(4)$$

Similarly $r_{cd} = m_c m_d$, and so on.

Hence

$$r_{ab}r_{cd} - r_{ac}r_{bd} = 0. \quad \dots\dots(5)$$

The left-hand side of this equation is called a *tetrad difference*, and the equation is called a *tetrad relation*. By permuting the letters a, b, c, d, we get three such relations, but only two are independent.

4. *Experimental result concerning mental tests.* We now come to an experimental fact of the greatest importance, relating to what are often called by the question-begging name of *intelligence tests*, which we prefer to call *mental tests*. These may be of many different kinds, such as requiring those tested to describe what is represented in a given picture, deal with a problematical situation, detect the absurdities in an anecdote, find the shortest way through a maze, give the word with the opposite meaning to a given word, complete a sentence with a missing word, and so forth. It is not necessary to ensure that these tests correspond to any preconceived ideas of whatever meaning we may choose to attach to the word intelligence, but as far as possible the tests should not require a knowledge of facts. We take K such tests, each applied to the same set of N persons, and thus obtain sets of numbers such as dealt with in paragraph 3. *It is an experimental fact that for sufficiently dis-*

similar mental tests all the tetrad differences are approximately zero (without the coefficients of correlation themselves being zero.)

5. *The general factor g and the specific factor s.* There is a mathematical theorem which is the converse of that proved in paragraph 3.

For an infinite set of tests A, B, ... whose tetrad differences all vanish (without the coefficients of correlation themselves vanishing) a mathematical function g may be defined, and also an infinite set of other functions s_a, s_b, ..., uncorrelated to each other and to g. The score obtained by any individual in any test is the sum of two parts, one proportional to his g and the other proportional to s peculiar to that test. Thus each individual has only one *g* (the *general factor*) but as many *s*'s (*specific factors*) as there are dissimilar tests. The proportion of *g* and *s* that enter into any test varies with the test. In classical ability the general factor preponderates, but in musical ability the specific. Arithmetical tests appear to involve *g* and an arithmetical *s*, geometrical tests *g* and a geometrical *s*. Thus what is called mathematical ability seems to consist of at least three uncorrelated elements.

Before entering into the somewhat lengthy details of the definition of these factors and the proof of the theorem, we shall state the properties of *g* and *s* that make them important in psychology.

6. *Experimental results concerning g and s.* It is found that *g* increases with age, at first rapidly, then more and more slowly up to about 15 or 16, and then it normally remains constant until extreme old age (unless affected by serious accident or disease). It is approximately unimprovable by education or training, and increases with age in the same proportion for different individuals, so that whoever has most of it at any one early age, say 11, will have most of it all his life. On the other hand, each *s* is capable of great improvement by education or training. They vary with age in a highly complicated manner. Improvement or deterioration may occur at any time, and may be affected considerably by change of method or incentive.

We are tempted to identify *g* with general intelligence, or general mental energy, or maturity, and the *s*'s with special ability, or the strength of a special part of the brain. But this is to embark upon speculations difficult to verify, whereas the object of the present treatment is to present facts free from hypotheses. For this reason we shall continue to use the symbols *g* and *s*, defined solely by the mathematical expressions given later.

7. *Lag and oscillation of g.* There is some evidence that each person has a characteristic value for his inertia or lag in the application of *g*. Similarly each has a characteristic difference in the amount and rate of the oscillation of *g* caused by the deterioration following fatigue and the recovery after rest.

8. *Extension to tests of character.* An attempt has been made to form judgments on various qualities of character of a number of persons, and to work out the correlation between these qualities. The results seem to indicate the existence of a general factor *w*,

which is something of the nature of balance of character or self-control. However, the possibilities of error in these calculations are considerably greater than in the case of mental tests.

9. *Analysis of the mathematical properties of g, if it exists.* As a preliminary to the formulation of our definition of g, we find relations that must be true if equations such as (2) and (3) of paragraph 3 hold good.

From (2), $\quad ag = m_a g^2 + n_a s_a g,$

hence, averaging, $\quad r_{ag} = m_a.$

Similarly $\quad r_{bg} = m_b.$

Hence, using (4), $\quad r_{ab} = m_a m_b = r_{ag} r_{bg}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$

Using this and similar results for r_{ac} and r_{bc},

$$(r_{ab} r_{ac}/r_{bc})^{\frac{1}{2}} = (r_{ag}^2 r_{bg} r_{cg}/r_{bg} r_{cg})^{\frac{1}{2}} = r_{ag}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Also from (2), $\quad a^2 = m_a^2 g^2 + 2 m_a n_a g s_a + n_a^2 s_a^2,$

so, averaging, $\quad 1 = m_a^2 + n_a^2,$

and $\quad n_a^2 = 1 - m_a^2 = 1 - r_{ag}^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$

Now choose multipliers w_a, w_b, ... such that the coefficient of correlation between g and the combined test $t = \Sigma w_a a$ may be a maximum, where t is to be in standard measure.

Averaging the relation $\quad tg = \Sigma w_a ag,$

we get $\quad r_{tg} = \Sigma w_a r_{ag}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$

Also averaging $\quad t^2 = \Sigma w_a^2 a^2 + 2 \Sigma\Sigma w_a w_b ab,$

we get $\quad 1 = \Sigma w_a^2 + 2 \Sigma\Sigma w_a w_b r_{ab};$

so, using (6), $\quad 1 = \Sigma w_a^2 + 2 \Sigma\Sigma w_a w_b r_{ag} r_{bg},$

i.e. $\quad 1 = (\Sigma w_a r_{ag})^2 + \Sigma\{w_a^2 (1 - r_{ag}^2)\}. \quad \ldots\ldots (10)$

From (9) and (10), $\quad 1/r_{tg}^2 = 1 + \Sigma\{w_a^2 (1 - r_{ag}^2)\}/(\Sigma w_a r_{ag})^2, \quad \ldots\ldots (11)$

so r_{tg} is greatest when $\Sigma\{w_a^2 (1 - r_{ag}^2)\}/(\Sigma w_a r_{ag})^2$ is least.

But from the identity

$$\{\Sigma w_a^2 (1 - r_{ag}^2)\} \left\{ \sum \frac{r_{ag}^2}{1 - r_{ag}^2} \right\} - \{\Sigma w_a r_{ag}\}^2$$

$$= \sum\sum \left\{ \frac{w_a r_{bg} (1 - r_{ag}^2)^{\frac{1}{2}}}{(1 - r_{bg}^2)^{\frac{1}{2}}} - \frac{w_b r_{ag} (1 - r_{bg}^2)^{\frac{1}{2}}}{(1 - r_{ag}^2)^{\frac{1}{2}}} \right\}^2,$$

of which the right-hand side is a sum of squares, it follows that the least value of either side is zero. Dividing each side by

$$\left\{ \sum \frac{r_{ag}^2}{1 - r_{ag}^2} \right\} \{\Sigma w_a r_{ag}\}^2,$$

which is positive, we see that the least value of

$$\Sigma\{w_a^2 (1 - r_{ag}^2)\}/(\Sigma w_a r_{ag})^2 \text{ is } 1 \Big/ \left\{ \sum \frac{r_{ag}^2}{1 - r_{ag}^2} \right\} = \frac{1}{S}, \text{ say,}$$

so from (11) the greatest value of r_{tg} is

$$(1 + 1/S)^{-\frac{1}{2}} = \{S/(1 + S)\}^{\frac{1}{2}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

This maximum is attained when

$$w_a(1-r_{ag}^2)/r_{ag} = w_b(1-r_{bg}^2)/r_{bg} = \ldots = \lambda, \text{ say}. \quad \ldots\ldots\ldots(13)$$

From (10), $$1 = \lambda^2\left[\left\{\sum \frac{r_{ag}^2}{1-r_{ag}^2}\right\}^2 + \sum \frac{r_{ag}^2}{1-r_{ag}^2}\right] = \lambda^2(S^2+S). \quad (14)$$

From (13), the greatest r_{tg} corresponds to $t = \lambda \sum \frac{ar_{ag}}{1-r_{ag}^2}$. If we apply the regression equation (1) of paragraph 2 with a and b replaced by t and g respectively, using $\sigma_t = \sigma_g = 1$ and equations (12) and (14), we get for the estimate of g

$$tr_{tg} = \left(\frac{S}{1+S}\right)^{\frac{1}{2}} \lambda \sum \frac{ar_{ag}}{1-r_{ag}^2} = \frac{1}{1+S} \sum \frac{ar_{ag}}{1-r_{ag}^2}. \quad \ldots\ldots\ldots(15)$$

Now put $$g - tr_{tg} = ki, \quad \ldots\ldots\ldots(16)$$

where k is a constant chosen to make $\sigma_i = 1$.

Squaring and averaging (16),

$$1 - 2r_{tg}^2 + r_{tg}^2 = k^2;$$

so from (12), $k^2 = 1/(1+S)$.

From this and (16), $$g = \frac{1}{1+S}\left(\sum \frac{ar_{ag}}{1-r_{ag}^2}\right) + \frac{i}{(1+S)^{\frac{1}{2}}}. \quad \ldots\ldots\ldots(17)$$

From (2), (8), and (17),

$$s_a = (1-r_{ag}^2)^{-\frac{1}{2}}\left\{a - \frac{r_{ag}}{1+S}\left(\sum \frac{ar_{ag}}{1-r_{ag}^2}\right) - \frac{ir_{ag}}{(1+S)^{\frac{1}{2}}}\right\}. \quad \ldots\ldots\ldots(18)$$

It is impossible to find the set of N numbers denoted by i, for the only conditions that they need satisfy (see the next paragraph) are in general insufficient to determine them. However, by taking N large enough the coefficients of i in (17) and (18) can be made as small as we please, thus giving in the limit definite expressions for g and s_a. Even a single test for which r_{ag} is nearly unity will make S large and so the coefficient of i small.

10. *Definition of g and s.* We now investigate the converse of the results in paragraphs 3 and 9. Starting from a number of tests satisfying the tetrad relations, it follows that

$$r_{ab}r_{ac}/r_{bc} = r_{ab}r_{ad}/r_{bd} = r_{ac}r_{ad}/r_{cd},$$

thus the value of $(r_{ab}r_{ac}/r_{bc})^{\frac{1}{2}}$ is independent of b and c, and depends on a alone, so may be denoted by α. Of course, it will turn out later that α is r_{ag}, but it is illogical to introduce a reference to g before it has been defined. Similarly we use β for what will later prove to be r_{bg}, and so on.

We now *define* g and s_a by what equations (17) and (18) become when r_{ag}, r_{bg}, ... are replaced by α, β, ... and where i denotes an indeterminate set of N numbers (reduced to standard measure, like a, b, ...), of which all that is postulated is that it is uncorrelated with any of a, b, ...

It will simplify the subsequent work to show that these definitions give g and s_a in standard measure.

Averaging the modified (17) and (18), we see that the means of g and s_a both vanish. Squaring the modified (17) and averaging, we get

$$\sigma_g^2 = \frac{1}{(1+S)^2}\left[\sum \frac{a^2}{(1-a^2)^2} + 2\sum\sum \frac{a\beta r_{ab}}{(1-a^2)(1-\beta^2)}\right] + \frac{1}{1+S}.$$

But
$$a\beta = (r_{ab}r_{ac}/r_{bc})^{\frac{1}{2}}(r_{ba}r_{bc}/r_{ac})^{\frac{1}{2}} = r_{ab}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

and

$$2\sum\sum \frac{a^2\beta^2}{(1-a^2)(1-\beta^2)} = \left(\sum \frac{a^2}{1-a^2}\right)^2 - \sum \frac{a^4}{(1-a^2)^2} = S^2 - \sum \frac{a^4}{(1-a^2)^2},$$

so
$$\sigma_g^2 = \frac{1}{(1+S)^2}\left[\sum \frac{a^2-a^4}{(1-a^2)^2} + S^2\right] + \frac{1}{1+S} = \frac{S+S^2}{(1+S)^2} + \frac{1}{1+S} = 1.$$

Similarly squaring the modified (18) and averaging, we get

$$\sigma_{s_a}^2 = (1-a^2)^{-1}\left[1 + \frac{a^2}{(1+S)^2}\left\{\sum \frac{a^2}{(1-a^2)^2} + 2\sum\sum \frac{a\beta r_{ab}}{(1-a^2)(1-\beta^2)}\right\} + \frac{a^2}{1+S} - \frac{2a}{1+S}\left\{\frac{a}{1-a^2} + \frac{\beta r_{ab}}{1-\beta^2} + \ldots\right\}\right]$$

$$= (1-a^2)^{-1}\left[1 + \frac{a^2}{(1+S)^2}\{S+S^2\} + \frac{a^2}{1+S} - \frac{2a^2}{1+S}\left\{\frac{1-a^2+a^2}{1-a^2} + \frac{\beta^2}{1-\beta^2} + \ldots\right\}\right]$$

$$= (1-a^2)^{-1}\left[1 + a^2 - \frac{2a^2}{1+S}(1+S)\right] = 1.$$

11. *Proof of the two-factor theorem.* Eliminating i from the modified (17) and (18), we get

$$a = ag + (1-a^2)^{\frac{1}{2}}s_a. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Squaring (20) and averaging,

$$1 = a^2 + 2a(1-a^2)^{\frac{1}{2}}r_{gs_a} + (1-a^2),$$

so
$$r_{gs_a} = 0. \quad \text{Similarly} \quad r_{gs_b} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

From (20) and a similar equation for b, by multiplication,

$$ab = a\beta g^2 + \beta(1-a^2)^{\frac{1}{2}}gs_a + a(1-\beta^2)^{\frac{1}{2}}gs_b + (1-a^2)^{\frac{1}{2}}(1-\beta^2)^{\frac{1}{2}}s_as_b,$$

whence, averaging and using (19) and (21),

$$r_{ab} = r_{ab} + (1-a^2)^{\frac{1}{2}}(1-\beta^2)^{\frac{1}{2}}r_{s_as_b},$$

so
$$r_{s_as_b} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

Similarly the coefficient of correlation between any other pair of s's vanishes.

By taking K (the number of tests) to be infinite we obtain the theorem enunciated in paragraph 5.

12. *Errors of sampling.* As already stated, the observed tetrad differences for dissimilar mental tests are *approximately* zero. It is possible that the observed non-zero values are to be attributed solely to the *error of sampling*, due to taking N too small. Considerable controversy has arisen on this and other parts of the

theory. Professor Spearman's ideas have passed through a process of evolution, and in their earlier forms they were certainly open to criticism. It is hoped that the proof given in paragraphs 9-11 is sound; I published an abstract of it in *Nature* (10th Jan. 1931) and it is accepted in subsequent papers by H. B. Heywood, J. O. Irwin, and J. C. Maxwell Garnett.

In the above account, intended for mathematicians rather than psychologists, many aspects of Professor Spearman's work have been omitted, such as his *Noegenetic laws*. For these reference should be made to his books, *The Nature of "Intelligence" and the Principles of Cognition* (1923) and *The Abilities of Man* (1927). The mathematical appendix to the latter work was, of course, written before the publication of the mathematical investigations of the last two years.

[Some new results in the two-factor theory, obtained after this article was in proof, have been presented to a psychological journal for publication under the title, *Three sets of conditions necessary for the existence of a g that is real and unique except in sign.*]

H. T. H. PIAGGIO.

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900. [Albert, first Duke of Prussia] Fond of the profane Sciences too, especially of Astronomy: Erasmus Reinhold and his *Tabulae Prutenicae* were once very celebrated; Erasmus Reinhold proclaims gratefully how these his elaborate Tables (done according to the latest discoveries, 1551 and onwards) were executed upon Duke Albert's high bounty; for which reason they are dedicated to Duke Albert, and called "*Prutenicae*", meaning *Prussian*. The University of Königsberg was already founded several years before, in 1544.—Carlyle, *Frederick the Great*, i. p. 181 (1872).

901. [April 1705.] The Queen knighted Isaac Newton, her Master of the Mint, who was prospective Whig candidate for the University, which he had on some former occasions represented in Parliament. But the High Churchmen were not to be thus overawed. At the election next month they spoke contemptuously of Newton as "a late made knight" and, according to the Whig Bishop of Ely, "it was shameful to see a hundred or more young students encouraged into hollowing like schoolboys and porters, and crying *No Fanatic, No Occasional Conformity* against two worthy gentlemen that stood candidates". The two "worthy gentlemen" were Newton himself and Godolphin's son Francis, who were handsomely beaten by a Tacker and another High Tory. The cry of "No fanatic" raised against Isaac Newton by a mob of young gentlemen about to take holy orders has its humorous side. Newton was right at the bottom of the poll, but he had consented to stand too late and too reluctantly and had refused to canvass; it was said that the majority of the residents had voted for Godolphin and Newton, but had been out-voted by the non-residents, who were for the most part country parsons.—G. M. Trevelyan, *England under Queen Anne: Ramillies and the Union with Scotland* (1932), pp. 28-29.

902. Newtonus in hoc tertio Libro telluris motæ hypothesim assumit. Autoris propositiones aliter explicari non poterant, nisi eâdem quoquè factâ hypothesi. Hinc alienam coacti sumus gerere personam. Cæterum latis à summis Pontificibus contrà telluris motum Decretis nos obsequi profitemur. Le Seur et Jacquier. Declaratio, prefixed to Book III in their edition of the *Principia*, 1742. [Per Prof. E. H. Neville.]

OPERATIONAL PROOFS OF SOME IDENTITIES.

By W. H. McCrea, M.A., Ph.D.

The interest of pure mathematicians is nowadays shifting from the study of special functions to general function theory. In fact the study of special functions might ultimately be forsaken, as for example in geometry the study of the triangle is now virtually a closed chapter, were it not for one important consideration. That is, that these functions must always be in constant use in mathematical physics. Hence their properties tend to become questions of practical application rather than of theoretical interest. It follows that there is a demand for some method, even if only a formal one, which will derive the required properties in an expeditious manner, without always calling for the services of the professional pure mathematician. The purpose of this note is to show how a beginning might be made by using a calculus of operators. We illustrate the method by obtaining some identities for functions of hypergeometric type.

For convenience of reference we adhere to the notation of some related work by W. O. Kermack and the writer.* If then q, Q are any variables, we denote by p, P respectively the operators $-\partial/\partial q$, $\partial/\partial Q$. We may then write formally

$$\phi(Q+A)=e^{AP}\phi(Q), \quad \psi(q+a)=e^{-ap}\psi(q), \quad \ldots\ldots\ldots\ldots(1)$$

where ϕ, ψ are any functions of Q, q respectively, and A, a are constants. This allows us to write any difference equation in operational form. We shall apply it, for example, to the Bessel function $J_n(t)$ regarded as a function of the order n, and write, say, $J_{Q+1}(q)=e^{P}J_Q(q)$.

Bessel Functions. The Bessel functions $J_n(q)$ are known to satisfy the recurrence relations

$$\frac{2n}{q}J_n(q)=J_{n-1}(q)+J_{n+1}(q), \quad \ldots\ldots\ldots\ldots(2)$$

$$J'_n(q)=\tfrac{1}{2}\{J_{n-1}(q)-J_{n+1}(q)\}. \quad \ldots\ldots\ldots\ldots(3)$$

We may write these in our notation as

$$qJ_Q(q)=2(e^{P}+e^{-P})^{-1}QJ_Q(q), \quad \ldots\ldots\ldots\ldots(4)$$

$$pJ_Q(q)=\tfrac{1}{2}(e^{P}-e^{-P})J_Q(q). \quad \ldots\ldots\ldots\ldots(5)$$

From (5) we have, if a is any constant,

$$e^{-ap}J_Q(q)=\exp\{-\tfrac{1}{2}a(e^{P}-e^{-P})\}J_Q(q)$$

$$=\sum_{m=-\infty}^{\infty}J_m(a)e^{-mP}J_Q(q)$$

* W. O. Kermack and W. H. McCrea, *Proc. Edinburgh Math. Soc.* (2) **2** (1931), 205, 220. To be referred to as Papers I and II. *Proc. Royal Soc. Edinburgh*, **51** (1931), 176. The asymmetry of sign in p, P was introduced to deal with contact transformations from q, p to Q, P and is not essential in the present note.

from the definition of the Bessel *coefficients* $J_m(a)$. Interpreting the exponentials by (1), we have the result

$$J_Q(q+a) = \sum_{m=-\infty}^{\infty} J_m(a) J_{Q-m}(q)$$

which is the known *addition formula for Bessel functions*. It is to be observed that it holds formally for *any* function satisfying the single recurrence relation (3).

From (4), $qe^{-P}J_Q(q) = 2(1+e^{2P})^{-1}QJ_Q(q)$
$= 2\{1 - e^{2P} + e^{4P} - \ldots\}QJ_Q(q)$,

giving the identity,* when we write n for Q,

$$qJ_{n-1}(q) = 2\{nJ_n(q) - (n+2)J_{n+2}(q) + (n+4)J_{n+4}(q) - \ldots\}.$$

From (5), $-\frac{1}{2}p^{-1}J_Q(q) = e^P(1-e^{2P})^{-1}J_Q(q)$
$= (e^P + e^{3P} + e^{5P} + \ldots)J_Q(q)$.

Clearly we must interpret p^{-1} as the operation of integration, so we have the identity, writing n for Q,

$$\tfrac{1}{2}\int_0^q J_n(t)\,dt = J_{n+1}(q) + J_{n+3}(q) + J_{n+5}(q) + \ldots .$$

The limits of integration are determined on the assumption $n \geqslant 0$, by considering the case $q = 0$.

Legendre Functions. Legendre functions $P_n(q)$ satisfy the recurrence relation

$$P'_{n+1}(q) - 2qP'_n(q) + P'_{n-1}(q) = P_n(q),$$

which we translate as

$$-(e^P - 2q + e^{-P})pP_Q(q) = P_Q(q),$$

or, writing $q = \cos\theta$,

$$-\frac{1}{\sin\theta}(e^P - e^{-i\theta})(1 - e^{-P+i\theta})\frac{d}{d\theta}P_Q(\cos\theta) = P_Q(\cos\theta).$$

We have therefore

$$-\frac{1}{\sin\theta}(e^P - e^{-i\theta})\frac{d}{d\theta}P_Q(\cos\theta) = (1 - e^{-P+i\theta})^{-1}P_Q(\cos\theta)$$
$$= (1 + e^{-P+i\theta} + e^{-2P+2i\theta} + \ldots)P_Q(\cos\theta).$$

Equating real and imaginary parts,

$$-\frac{1}{\sin\theta}(e^P - \cos\theta)\frac{d}{d\theta}P_Q(\cos\theta)$$
$$= (1 + \cos\theta\, e^{-P} + \cos 2\theta\, e^{-2P} + \ldots)P_Q(\cos\theta)$$
$$-\frac{d}{d\theta}P_Q(\cos\theta) = (\sin\theta\, e^{-P} + \sin 2\theta\, e^{-2P} + \ldots)P_Q(\cos\theta).$$

* T. M. MacRobert, *Functions of a Complex Variable* (1925), 253, Ex. 13.

Taking for simplicity $Q = n$, a positive integer, and recalling that Legendre coefficients of negative order are zero, we have the identities :

$$-\operatorname{cosec}\theta\frac{d}{d\theta}P_{n+1}(\cos\theta)+\cot\theta\frac{d}{d\theta}P_n(\cos\theta)=P_n(\cos\theta)$$
$$+\cos\theta\,P_{n-1}(\cos\theta)+\cos 2\theta\,P_{n-2}(\cos\theta)+\ldots+\cos n\theta\,P_0(\cos\theta),$$
$$-\frac{d}{d\theta}P_n(\cos\theta)=\sin\theta\,P_{n-1}(\cos\theta)+\sin 2\theta\,P_{n-2}(\cos\theta)+\ldots$$
$$+\sin n\theta\,P_0(\cos\theta).$$

Parabolic Cylinder Functions. A recurrence relation * satisfied by the parabolic cylinder functions $D_n(q)$ is

$$D'_n(q)-\tfrac{1}{2}qD_n(q)+D_{n+1}(q)=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

which we write as

$$(p+\tfrac{1}{2}q)D_Q(q)=e^PD_Q(q).$$

This gives, where a is any constant,

$$\exp\{-a(p+\tfrac{1}{2}q)\}D_Q(q)=\exp(-ae^P)D_Q(q). \quad \ldots\ldots\ldots\ldots(7)$$

Now the operator $\exp\{-a(p+\frac{1}{2}q)\}$, which represents the series

$$1-a(p+\tfrac{1}{2}q)+\frac{a^2}{2!}(p+\tfrac{1}{2}q)(p+\tfrac{1}{2}q)-\frac{a^3}{3!}(p+\tfrac{1}{2}q)(p+\tfrac{1}{2}q)(p+\tfrac{1}{2}q)+\ldots,$$

is not easy to interpret directly, on account of the non-commutative character of q, p. But it can be evaluated by use of a theorem of Kermack and the writer (Paper II, Theorem IX), which gives

$$e^{-\frac{a}{2}(q+2p)}=e^{-\frac{a^3}{4}}e^{-\frac{a}{2}q}e^{-ap}.$$

We have therefore, from (7),

$$e^{-\frac{a^3}{4}}e^{-\frac{a}{2}q}e^{-ap}D_Q(q)=\left(1-ae^P+\frac{a^2}{2!}e^{2P}-\ldots\right)D_Q(q).$$

Hence we have the general *addition formula*, writing n for Q, formally true for any function satisfying the recurrence relation (6) :

$$e^{-\frac{a}{2}q-\frac{a^3}{4}}D_n(q+a)=D_n(q)-aD_{n+1}(q)+\frac{a^2}{2!}D_{n+2}(q)-\frac{a^3}{3!}D_{n+3}(q)+\ldots.$$

Conclusion. The object has been to exhibit a method, not to claim importance for the results. The identities given can all be *verified* without difficulty by direct means. The point illustrated, however, is that they can all be *discovered* by the single method of transforming operators. W. H. M'C.

* Whittaker and Watson, *Modern Analysis* (1927), 350.

TRANSPORTABILITY OF A SUNDIAL.

By Prof. D. M. Y. Sommerville.

It does not seem to be generally known that a sundial, which has been correctly constructed and graduated for a particular latitude, can be used in any other latitude if it is set up at the proper angle. This appears from the very simple observation that if a sundial is moved in any way without rotation (of course within a range small compared with the sun's distance), the shadow of the style will continue to fall upon the same figure of the dial; and if it remains on the same half-meridian of the Earth, where the local time is the same, the dial time will be still the same; when transported to another meridian, without change of latitude, it will give the correct local solar time for that meridian. (In transporting to another meridian, the dial will of course be rotated, so that its inclination and orientation relative to the earth's surface shall be unaltered.) Thus a horizontal sundial constructed for north latitude ϕ will be accurate when erected as a prime-vertical sundial, facing north, in south latitude $90° - \phi$ (Fig. 1).

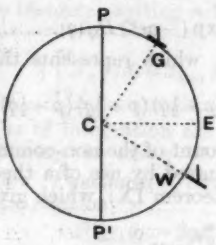

Fig. 1.

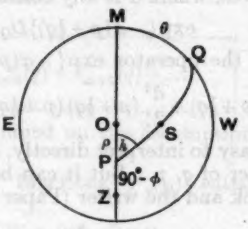

Fig. 2.

This appears also from the equation which connects the sun's hour-angle h with the angle θ which the edge of the shadow makes with the noon-line of the dial. Consider first the general "symmetrical" sundial whose plane is perpendicular to the plane of the meridian. Let Fig. 2 represent the projection of the celestial sphere on the plane of the dial, the centre O of the circle being the pole of this plane, or the normal to the dial; Z is the zenith, P the (north) pole, and S the sun. $MPS = h$, the sun's hour-angle west; $PZ = 90° - \phi$, the co-latitude; and $PO = \rho$, the polar-distance of O. SP represents the plane through the sun and the style (since the style is parallel to the polar axis), and Q represents the line of intersection of this plane with the plane of the dial, i.e. the edge of the shadow on the dial; PM is the meridian; $MQ = \theta$, the angle which the edge of the shadow makes with the noon-line OM. Then in the right-angled spherical triangle MPQ, where $MP = 90° + \rho$,

$$\tan \theta = \cos \rho \tan h.$$

For a given sundial the angle $90° - \rho$, which the style makes with the plane of the dial, is fixed, hence the relation between θ and h is fixed and independent of the latitude.

As an example, suppose we have a horizontal sundial which has been constructed for Greenwich (lat. 51° 28′ N.), and we wish to erect it at Wellington, N.Z. (lat. 41° 17′ S.). In Fig. 1, $ECG = 51° 28'$ and $ECW = 41° 17'$, hence $GCW = 92° 45'$, and the plane of the dial is inclined downwards to the north from the prime vertical at an angle 2° 45′ ; the style points downwards to the north pole.

For a horizontal dial $\rho = 90° - \phi$ and $\tan \theta = \sin \phi \tan h$. In the northern hemisphere θ and h are of the same sign, and the direction of motion of the shadow is clockwise. In the southern hemisphere ϕ, and therefore θ, is negative, and the direction of motion of the shadow is counterclockwise ; in this case the style points in the opposite direction.

For a prime-vertical dial facing north $\rho = \phi$ and $\tan \theta = \cos \phi \tan h$. The shadow, when it exists, both in the northern and the southern hemisphere, moves clockwise, but in the northern hemisphere a north-facing dial is of little use. On the other hand, in a south-facing prime-vertical dial the shadow, in both hemispheres, moves in the counterclockwise direction.

The simple rule is that the shadow moves clockwise when the style is directed to the north pole, counterclockwise when it is directed to the south pole.

An oblique dial, whose plane is not perpendicular to the meridian, is really one which is constructed for the wrong meridian. Let the pole of the dial-plane have north-polar distance ρ and west hour-angle η. If the sundial is transported without rotation to a place in latitude $90° - \rho$ and relative longitude to the west η, it becomes a horizontal dial for that place, but the time indicated by the dial will be the local solar time for relative longitude η east, and will be that much fast, *i.e.* the central graduation will be η instead of XII.

D. M. Y. Sommerville.

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903. "The poor League of Nations which . . . must escape from its impossible task by applying neutral formulas upon the basis of the lowest common denominator, which, as one knows, is often zero."—*The Times*, 6th Oct. 1932 ; quoted by the *Times* Rome correspondent from an attack made upon the Lytton Report in the *Mattino*. [Per both Mr. F. P. White and Mr. R. M. Wright.]

904. . . . and the physical philosopher. A word is much wanted for this creature. *Physical philosopher* is unobjectionable, but too long. *Philosopher*, by itself, a word which has been used, is not only proved to be too wide, but, in the revival of philosophy proper which is taking place, will go near to be thought so shortly. *Physician* has irrevocably lost its original sense ; and so has *naturalist*. *Physicist* is coming into use : it was a word invented in Pandemonium as a salute to Satan when he returned from *his* enquiry into nature, undertaken with a view of creating confusion by the introduction of his own notions ; and Milton has made the mistake of supposing that he was *hissed*. . . . —De Morgan, *Journ. Inst. Actuaries*, x. (1862), p. 248. [Per Dr. G. J. Lidstone.]

ANGLES OF PEDAL TRIANGLES.

By C. O. Tuckey.

The connection between the fact that $2^{12}-1$ is a multiple of 45 and the angles of pedal triangles may not be obvious but it is there, as Euclid may have remarked when chaffed about his suggestion of a liaison between medial section and the angles of an isosceles triangle.

If the angles of a triangle are A, B, C, the angles of the pedal triangle are $180°-2A$, $180°-2B$, $180°-2C$ if A, B, C are acute, but if A is obtuse they are $2A-180°$, $2B$, $2C$. Hence if you start with angles of a triangle, each of which is a whole number of degrees; form the angles of the pedal triangle, then the angles of the pedal triangle of that, and go on repeating the process, you will find that you keep switching about from one law of formation to the other, and will feel that something is bound to happen. What happens was pointed out to me by Mr. N. J. Chignell, but before stating his result in words it will be well to give a couple of examples.

Starting with			or with		
47,	53,	80	73,	11,	96
we get in succession :			we get :		
86,	74,	20	146,	22,	12
(*A*) 8,	32,	140	112,	44,	24
16,	64,	100	44,	88,	48
32,	128,	20	92,	4,	84
64,	76,	40	4,	8,	168
52,	28,	80	8,	16,	156
104,	56,	20	16,	32,	132
28,	112,	40	32,	64,	84
56,	44,	80	116,	52,	12
68,	92,	20	52,	104,	24
136,	4,	40	104,	28,	48
92,	8,	80	28,	56,	96
4,	16,	160	56,	112,	12
(*A*) 8,	32,	140	112,	44,	24

If any reader is interested he should at this point work out a few sets for himself. He will find that

(i) recurrence begins as soon as each angle is a multiple of 4° (which happens not later than the third triangle) ;

(ii) recurrence is in sets of 12 triangles, or in special cases in some sub-multiple of 12 ;

(iii) there is considerable variety in the sequences (for instance, consider what precedes and follows 8, 16, 32 in columns 1, 2, 4, 5 above).

Now for the reasons. Suppose we start when each angle is a multiple of 4° and that the angles are then $\alpha°$, $\beta°$, $\gamma°$. Following α we must have either $2\alpha-180$ if α is obtuse, or 2α if β or γ is obtuse, or $180-2\alpha$ if the triangle is acute-angled, in which case β is followed by $180-2\beta$ and γ by $180-2\gamma$. Thus supposing the formulae used to be (say) 1st, 2nd, 2nd, 3rd, 2nd, 1st in this order, the results

will be $2\alpha - 180$, $4\alpha - 2.180$, $8\alpha - 4.180$, $9.180 - 16\alpha$, $18.180 - 32\alpha$, $35.180 - 64\alpha$. Hence it is not hard to see that at the twelfth stage after α, the corresponding angle will be either $180n - 2^{12}\alpha$ or $2^{12}\alpha - 180s$, where n and s are integers. Further, the former case will arise if there has been an *odd* number of *acute*-angled triangles in the sequence, and the latter if there has been an *even* number. So that, at the twelfth stage, the three angles are either

$$180n_1 - 2^{12}\alpha,\ 180n_2 - 2^{12}\beta,\ 180n_3 - 2^{12}\gamma, \quad \text{.....(I)}$$

or

$$2^{12}\alpha - 180s_1,\ 2^{12}\beta - 180s_2,\ 2^{12}\gamma - 180s_3. \quad \text{.....(II)}$$

It will be shown that (I) is impossible, so let us start with (II). Consider the *differences* between the angles obtained and the original ones. The first of these is $2^{12}\alpha - 180s_1 - \alpha$ or $(2^{12} - 1)\alpha - 180s_1$. Now $2^{12} - 1$ is a multiple of 45 and α is a multiple of 4, and so $(2^{12} - 1)\alpha - 180s_1$ is a multiple of 180. But this expression is the difference between two angles, each of which is less than 180. Thus the expression is both less than 180 and a multiple of 180. It is therefore zero, and similarly for the other two angles. Thus in (II) —that is, if there is an even number of acute-angled triangles in the sequence of twelve—the angles of the twelfth set after α, β, γ are again α, β, γ.

Now take (I) and consider the *sums* of the angles obtained and the original ones. The first is $180n_1 - 2^{12}\alpha + \alpha$, or $180n_1 - (2^{12} - 1)\alpha$, which as before is a multiple of 180; but it is the sum of two angles each of which is less than 180, therefore this sum must be 180 itself, and similarly for the other two angles. Thus if there is an odd number of acute-angled triangles in the sequence of twelve, the angles of the twelfth are $180 - \alpha$, $180 - \beta$, $180 - \gamma$. But the sum of these three angles is 360°, therefore this case is impossible.

Hence of the twelve triangles an *even* number must be *acute-angled*, and the angles of the twelfth set after α, β, γ are again α, β, γ.

This result is the main object of this discussion; there are one or two minor points on which I add notes:

(i) If α is a multiple of 3, all subsequent angles got from it will also be multiples of 3. The same applies if α is a multiple of 5 or of 9.

(ii) Hence if α, β, γ are all multiples of 12, since $2^4 - 1 = 15$ the recurrence will be in sets of four triangles.

(iii) If α, β, γ are all multiples of 20, since $2^3 + 1 = 9$, there will be recurrence in sets of three triangles, one acute-angled.

(iv) If α, β, γ are multiples of 36, since $2^2 + 1 = 5$, the recurring period consists of two triangles.

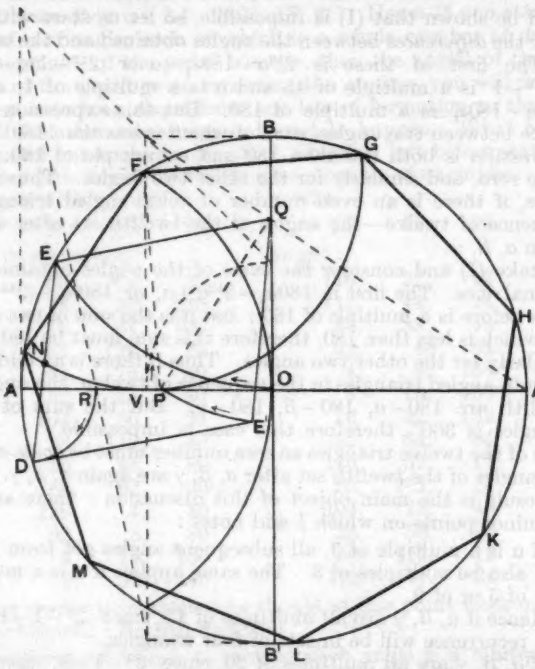
(v) The multiples of 4 which are less than 180 and are not multiples of 3 or 5 number 24, being twelve angles and their twelve supplements. The sequences following one of these will contain half of the twenty-four angles, not always the same twelve, but selected so that in the twelve angles two supplementary angles do not occur. This is because $2^r \pm 2^s$ is not a multiple of 45 for values of r and s less than 12.

C. O. TUCKEY.

MATHEMATICAL NOTES.

1062. ***To inscribe a regular heptagon in a circle, so that the error for the side shall be less than one-millionth of the radius.***

Let O be the centre of the circle, and AA', BB' any two diameters perpendicular to each other. Trisect the arc AB in E and F, and bisect AO in V. Divide AV at R in the ratio 3 : 2, and divide OB at Q in the ratio 2 : 1. (The dotted lines in the figure merely suggest how R and Q may be most simply and accurately obtained.)



With centre A and radius AR describe an arc cutting the arc AB' in D. Join DE, EQ and complete the parallelogram $DEQE'$. With centre Q and radius QE' describe an arc cutting OA in P.

Then FP is approximately equal to the side of a regular heptagon inscribed in the circle, the error (in excess) being about ·0000007 AO.

For, denoting AO by 1, $AR = \cdot 3$, and the angle DOA is $17° 15\frac{663}{2876}'$. But the angle AOE is 30°. Hence DE is the chord of

$$47° 15\tfrac{663}{2876}' = \cdot 8015591.$$

But $QP = QE' = DE$; hence $QP^2 = \cdot 64249701951$, and $QO^2 = \cdot 4$. Hence $OP^2 = \cdot 19805257506\ldots$, and $OP = \cdot 4450309\ldots$. But $OV = \cdot 5$,

and so $PV = \cdot 054969\ldots$, $PV^2 = \cdot 0030216\ldots$. Also $FV^2 = \cdot 75$, and thus $FP^2 = \cdot 7530216\ldots$, $FP = \cdot 86776817\ldots$. Now the side of a regular heptagon inscribed in the unit circle is $\cdot 86776751\ldots$. Thus the error is about $\cdot 0000007$ at most. Hence if equal chords FG, GH, HK were drawn towards A', and similar chords FN, NM, ML towards A, the error (in defect) for the final chord KL would be less than $\cdot 000005\ AO$, that is, less than an inch for a circle six miles in diameter!

G. WOTHERSPOON.

1063. *Prime Numbers.*

Dr. Gaetano Fazzari's Arithmetic ascribes to Euler the observation that

$x^2 + x + 17$ is prime for all integral values of x from 0 to 15,

$x^2 + x + 41$ is prime for all integral values of x from 0 to 39,

and to Legendre that

$2x^2 + 29$ is prime for all integral values of x from 0 to 28.

C. Smith's *Treatise on Algebra*, p. 481, ascribes the second of these to Euler, the others to Barlow.

Another fact of the same sort, apparently not hitherto noticed, was brought to me by a pupil, R. H. Walton, that

$x^2 - 13x - 1$ is prime for all integral values of x from 14 to 24,

$13x - x^2 + 1$ is prime for all integral values of x from 0 to 13.

F. C. BOON.

1064. *On Mr. R. J. A. Barnard's article* (XVI, 220, pp. 254-5).

Mr. Barnard, in explaining his first method for solving differential equations, does not point out that it is unnecessary to remove the second term in

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = X \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

by a substitution, although it may be useful to do this in a preliminary explanation for beginners. Any solution of

$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0$$

is an integrating factor for (1); two such solutions being used in turn, dy/dx can be eliminated from the resulting equations, as in Mr. Barnard's method.

A similar treatment of the linear equation of order n, with constant coefficients,

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_ny = X$$

is possible, n independent solutions of

$$\frac{d^ny}{dx^n} - a_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + (-)^n a_ny = 0$$

being used in turn as integrating factors, and all the $n-1$ successive differential coefficients of y being eliminated from the n equations thus found.

F. E. CAVE.

1065. *On Note* 1046.

The definition of relative velocity in Mr. D. A. Young's note 1046, "The velocity of a point P relative to a point Q is the velocity with which P appears to move when viewed from Q", is open to serious objection. It might be considered frivolous to ask whether two points with an opaque object between them are incapable of having a relative velocity because neither can be viewed from the other. But it is well known that when a motor car is going at high speed the velocity with which a point of it appears to move when viewed by the driver is not usually zero; and the velocity with which a star appears to move, when viewed from the earth, may differ greatly from the relative velocity. The property actually used by Mr. Young is: "If we compound with the velocity of each the reversed velocity of Q, we bring Q to rest without altering the velocity of P relative to Q"; but it is not clear how this property follows from his definition.

F. E. CAVE.

1066. *Successive approximations to* $\sqrt[n]{a}$.

1. In Gleaning 640 (*Gazette*, XIV, January 1929, p. 308), a rule is given from the preface to Reuben Burrows' *The Theory of Gunnery*, which is equivalent to the following: if x_1 is an approximate value of $\sqrt[3]{a}$, then a better approximation is

$$x_2 = \frac{x_1^4 + 2ax_1}{2x_1^3 + a}.$$

We have, in fact, if a is the root to which we are approximating,

$$x_2 - a = \frac{(x_1 + a)(x_1 - a)^3}{2x_1^3 + a},$$

or, writing $$d_r = x_r - a,$$

$$d_{r+1} = \frac{x_r + a}{2x_r^3 + a} \cdot d_r^3.$$

It is easy to prove that if h_r is the smaller of x_r, a, then

$$d_{r+1} < d_r^3/h_r^2.$$

Hence if $h_r > d_r$, that is, if x_r lies between $\frac{1}{2}a$ and $2a$, a better approximation is obtained. Again, if g_r is the greater of x_r, a, we can prove that

$$d_{r+1} > d_r^3/2g_r^2.$$

When our approximations are "close" we have, approximately,

$$d_{r+1} = \frac{2a}{3a} \cdot d_r^3,$$

or, in words, any error is approximately proportional to the cube of the preceding error.

2. We now proceed to obtain a formula giving successive approximations to $\sqrt[n]{a}$, possessing this property; n is a positive integer

greater than 1, and we denote $\sqrt[n]{a}$ by a. We assume, k being a constant depending on n, that the approximation is of the form

$$x_{r+1} = \frac{x_r^{n+1} + kax_r}{kx_r^n + a} \quad \text{.....(i)}$$

Writing $d_r = x_r - a$, $D_r = kx_r^n + a$, we have

$$d_{r+1} = d_r[x_r^n + (1-k)x_r^{n-1}a + \ldots + (1-k)x_r a^{n-1} + a^n]/D_r \quad \text{....(ii)}$$

where in the bracket there are $(n+1)$ terms, the unwritten coefficients being each $1-k$. We now choose k so that the expression in square brackets is divisible by d_r; since $d_r = x_r - a$, by the remainder theorem the sum of the coefficients must vanish, and so

$$2 + (n-1)(1-k) = 0,$$

whence $k = (n+1)/(n-1)$.

Now the derivative of this expression with respect to x_r, when tested in the same way for divisibility by $x_r - a$, gives the condition

$$n + (1-k)\{(n-1) + (n-2) + \ldots + 1\} = 0$$

or

$$n + \tfrac{1}{2}(1-k)n(n-1) = 0,$$

which again gives $k = (n+1)/(n-1)$. Thus for this value of k, the expression in square brackets contains a factor d_r^2, and (ii) now becomes

$$d_{r+1} = d_r^3[(n-1)x_r^{n-2} + 2(n-2)x_r^{n-3}a + 3(n-3)x_r^{n-4}a^2 + \ldots + (n-1)a^{n-2}]/(n-1)D_r.$$

When our approximations are "close", we have approximately

$$d_{r+1} = \frac{(n^2-1)\,d_r^3}{12a^2}.$$

The recurrence formula for the successive approximations is

$$x_{r+1} = \frac{(n-1)x_r^{n+1} + (n+1)ax_r}{(n+1)x_r^n + (n-1)a} \quad \text{.....(iii)}$$

The Grammar School, Hampton, Middlesex. G. W. WARD.

1067. *On Notes* 1042, 1059.

In Note 1042 Mr. F. J. D. Syer gave a solution of a certain set of equations. In Note 1059 Mr. F. C. Boon pointed out a considerably shorter solution. After the appearance of the first note, Mr. Syer noticed that the shorter solution was possible and sent it to the editor with a request that it might find a place in the *Gazette*, but owing to the time taken by a letter from Australia, his solution was not received until after the publication of the December *Gazette* containing Mr. Boon's note.

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905. Mathematics, rightly viewed, possesses not only truth, but also supreme beauty—a beauty cold and austere, like that of sculpture. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.—Bertrand Russell; quoted by Havelock Ellis, *The Dance of Life*, p. 128. [Per Mr. E. G. Hogg.]

REVIEWS.

Fastperiodische Funktionen. By HARALD BOHR. Pp. ii, 96. RM. 11.40. 1932. Ergebnisse der Mathematik, Band I, Heft 5. (Springer, Berlin)

Within a few months after my review of Mr. Besicovitch's book on *Almost Periodic Functions*, I find myself requested by the editor of the *Mathematical Gazette* to review this very similar treatise by Professor Harald Bohr.

The scope of the two books is almost identical, although Mr. Besicovitch naturally gives more space to the generalization of almost periodic functions—a field in which much of the basic work is due to Mr. Besicovitch himself. Both writers devote appendices to the almost periodic functions of a complex variable. Professor Bohr introduces more introductory material, including an excellent discussion of the ordinary theory of orthogonal functions and of the more elementary parts of Fourier Series.

The difference between the two books is mainly conditioned by the fact that whereas Mr. Besicovitch's book is the elaboration of a prize essay, Professor Bohr's treatise represents the final form of a series of lectures which was held at Göttingen some years ago and was there worked out by Mr. Schwerdtfeger, and which were repeated in a much more complete form at various American universities in 1930-31. Thus Professor Bohr's book is distinguished by being much more adaptable to pedagogical purposes and is much easier reading.

As is to be expected when two mathematicians have worked so much in collaboration, the methods chosen by the two authors are extremely similar. In particular, both base the proof of the fundamental theorem on the method of Professor de la Vallée Poussin. This method, as I have said in my previous review, is without any doubt as simple as any. Nevertheless, I cannot help feeling that the criticism which Professor Bohr directs against Professor Weyl's methods on the score that they involve a difficult reference to the theory of integral equations, is not altogether justified.

The approximately simultaneous appearance of the books of Bohr and Besicovitch may be taken as a sign that the theory of almost periodic functions is approaching its definitive form and will soon be a classical part of mathematical education. The great disadvantage of the theory as it stands at present is the somewhat scanty list of applications in physics and in analysis. Let us hope that so beautiful a theory will soon come into its own in this as in other respects.

NORBERT WIENER.

Einfachste Grundbegriffe der Topologie. By P. ALEXANDROFF. Pp. 48. RM. 3.60. 1932. (Springer, Berlin)

There are times when the average mathematician feels the desire to learn something of the work of other mathematicians, whose researches have been on lines unfamiliar to him, and it is not always easy to find literature suitable for this purpose. There must be many mathematicians whose ideas of topology are very vague but who would like to know the elements of the subject. Until the present publication appeared this has been impossible without spending more time than the person who is not a specialist in topology can usually afford, but we think this little book by Professor Alexandroff will provide the means of acquiring the desired knowledge. It has grown out of what was originally intended to be an appendix to *Vorlesungen über Anschauliche Geometrie* by Hilbert and Cohn-Vossen. Its purpose is to explain the objects with which the topologist is concerned and the methods by which he investigates the properties of these objects. As the book is itself a summary, we refrain from attempting a summary of its contents. It is sufficient to say that it is concerned with the presentation of the foundations of topology in as simple a light as possible, and serves as an excellent introduction to the subject. Its interest is for the general mathematician, and it does not attempt

to provide a detailed account even of those elementary portions of the subject which are often all that is required in the applications which can be made of topology to numerous and varied branches of mathematics. But all who feel any interest in topology—mercenary or otherwise—will find much in this little volume which will please them. W. V. D. H.

The Universe of Science. By H. Levy. Pp. xiv, 224. 7s. 6d. 1932. (Watts)

The emphasis of Professor Levy's book is on the one-ness of the universe. Not less than the cigarette which is being smoked, the table—why is it always a table ?—and the writer himself lose the clear-cut boundary that is usually thought to separate them as individuals from their environment. Isolation is artificial, but it is only of isolates that scientific laws are discoverable. A simple law being approximately satisfied, science aims not at the discovery of a more complex law to be satisfied exactly, since exactitude except in sheer enumeration is impossible, but at the discovery of the isolated system in which the simple law would be true. In the limit the process of isolation leads to the abstractions of physics and mathematics.

Mathematics introduced, Professor Levy illustrates the nature of mathematical proofs and definitions by examples analysed elaborately into their simplest steps. The view which he accepts, that the essence of mathematics is the rearrangement of symbols, is, it need hardly be said, only one, and not the easiest to defend, of the theories of mathematics which have been proposed. But Professor Levy throughout his book is concerned rather to expound one consistent scheme than to argue against others which will not be in the reader's mind, and it is hardly common knowledge that the nature of mathematics is the subject of difficult philosophical enquiry.

With determinism and the nature of time, topics of special interest to members of the Mathematical Association since Sir Arthur Eddington devoted his two Presidential Addresses to them, the case is quite different. Unlike Hilbert and Wittgenstein, Eddington, Jeans and Smuts bear names which are household words, and a writer on the philosophy of science cannot contradict these famous authorities without knowing that his readers will be aware of his temerity. Eddington's style in his popular writing delivers him almost defenceless into the hands of any critic who points out the literal implications of his more arresting sentences. Can it be true that an experimenter, unless it was entropy that he was measuring, does not know whether he took his observations before or after he set up his apparatus ? If Eddington is right in saying that " in our probing of the material universe we cannot find a particle of evidence in favour of determinism ", the greater part of his own life's work, concerned as it has been with enlarging the sphere of validity of scientific prediction, is groundless. So at least Eddington does not mean what he says, and an alternative theory has not to refute his assertions but only to account for the state of mind in which they were made. According to Professor Levy, the incompatibility between particle and wave dynamics, the notorious principle of uncertainty, and all such difficulties, are due to an overloading of isolates or symbols with a weight which there has never been the slightest reason to suppose them capable of carrying. It is always as if, having found that 2 sheep and 3 sheep are 5 sheep, we inferred that 6 half-carcases are 3 live animals. The mathematical physicist, forgetting that isolated electrons are beyond the range of direct observation, creates from their presumed properties a fictitious universe. " The breakdown on the smallest scale in the usual form of prediction, which to the experimenter is evidence that he has reached the limits of his possible technique, is to " the mathematician " the starting-point from which he will now proceed to evolve a large-scale universe, . . . with all the attributes of indeterminism he has inserted in it." But the starting-point of public reality is the actual large-scale universe, and the

determinism there, on which the meaning of science and the conduct of daily life depend, is not to be destroyed by hypothetical analysis and reconstruction.

In the last chapter, "Science—A social venture", Professor Levy considers, among other matters, the relation of the scientist to his environment; the scientist cannot work except on the basis of what is known to his contemporaries, and his results must command assent from those qualified to understand them. Professor Levy dismisses one characterisation of scientific aims on the ground that it excludes Rutherford from the company of scientists; he must be careful that right of entry is not denied to Mendel, tilling virgin soil in obscurity, or to Aristarchus, propounding a heliocentric theory more than eighteen hundred years before it was acceptable.

The book is full of good things, well expressed; it will stimulate discussion and provoke opposition. A slip in arithmetic on p. 114 and one in mechanics on p. 205 may be taken as showing how completely for the moment the mathematician in Professor Levy has given the word to the thoughtful and educated citizen, deeply interested in the social problems which are the common concern of us all.

E. H. N.

Differentialgeometrie. By L. Bieberbach. Pp. vi, 140. RM. 6. 1932. Teubners Mathematische Leitfäden, 31. (Teubner)

"Ich hoffe weiter, dass man in meinem Buche keine saloppen Gedankengänge finden wird, keine unsauberen Schlüsse von der Art, wie sie auch die moderne Literatur über Differentialgeometrie leider so oft noch beherbergt—als Überreste aus der Plüschmöbelzeit und als Verstandesschoner." It will be readily believed that these words went straight to my heart, or rather, since that is the whole point, to my head. An ambiguity in the last sentence on p. 54, and the assumption, surely unwarrantable, that the reader has a logically grounded familiarity with the fundamental ideas of connectivity, are the only lapses I noticed from the high ideal, while on the other side are not merely such care as is to be expected in dealing with contact, with geodesics, and with applicability, but a scrupulous calculation of the angular integral required in the Gauss-Bonnet theorem that will come as a revelation to the student even of Darboux.

At last we have a book in which curvature itself, not a radius of curvature, is recognised as fundamental. It is disappointing to find the curvature of a twisted curve required, in explicit contrast to that of a plane curve, to be positive by definition, "since we have no other way of fixing the corresponding unit vector". Professor Bieberbach accepts the implications of his convention, and presently his formulae are awkward, precisely because he comes to deal with cases in which there *are* other ways of fixing the vector. For example, if a unit vector is normal to a surface, its positive direction, chosen at one point, can be rendered definite by continuity throughout a region including that point; a curve for which the principal normal is normal to the surface can then have the sign of its curvature determined significantly by reference to this unit vector. The simplest formulation of Meusnier's theorem, instead of being inadequate, as of course it is in careless hands, remains valid to express just as much as Professor Bieberbach's carefully interpreted $\kappa \cos\theta = \pm K$.

By ignoring the concept of geodesic torsion Professor Bieberbach misses something of the structure of a surface, and in particular he is led to represent the Beltrami-Enneper expression for the torsion of an asymptotic line as a much less trivial result than it really is; it is perhaps for this reason that he finds this formula more interesting than formulae connected with the curvature of the line, but the latter formulae can hardly be understood without a knowledge of Laguerre's cubic function of direction and this itself involves the ignored concept. It may be said that in a small book an author must omit a great deal which he ought not to be accused of ignoring, but this is not a valid

defence when it is an opportunity to coordinate and thereby to compress a whole mass of material that has been lost.

However, when we realise how much Professor Bieberbach has contrived to include, dissatisfaction gives way to amazed enjoyment. The chapter on plane curves includes a proof of the theorem that a closed convex curve whose curvature is continuous cannot have fewer than four points of stationary curvature, and ends with a section on the curious problem of regions of constant breadth. An account of Levi-Cività's definition of parallel displacement brings the conception of a geodesic into line with the requirements of Riemannian geometry. There are sections on surfaces of constant curvature and on minimal surfaces. The elements of tensor calculus are so far developed as to show the nature of affine connection and the essential property of a Riemannian space. And there are five pages of formulae.

Surely to criticise details is ungracious. No one would have believed that such range and accuracy could be combined in so small a space if Professor Bieberbach had not accomplished the feat. E. H. N.

Analytische Geometrie. By L. Bieberbach. 2te Auflage. Pp. vi, 142. RM. 5.95. 1932. Teubners Mathematische Leitfäden, 29. (Teubner)

So short a time has elapsed since the first appearance of this booklet that we can reasonably refer our readers to the earlier review (Vol. XV, p. 271). For the new edition the type has been reset throughout, and paragraph headings have been added. The text is in the main unaltered, but a few paragraphs have been inserted, there are some more examples, and there is a concluding section on groups and the Erlanger programme which was not in the first edition, although reference was made to it in the companion volume on projective geometry published last year.

This work is not a substitute for the ordinary schoolbooks. It is concerned with the fundamental ideas of analytical geometry, not with the processes, however important, by which conics are submitted to analytical treatment. For example, nothing is said of families of conics, and we have to bear this in mind when we read that in affine geometry the number of distinguishable conics is finite; although an individual ellipse can have no affine properties that do not belong to a circle, two ellipses cannot be simultaneously reduced to circles by an affine transformation, and the affine geometry of a pair of ellipses is not that of a pair of circles.

It is a great satisfaction to welcome already this revised edition and to congratulate the author on his well-deserved success. E. H. N.

Einführung in die analytische Geometrie und Algebra. I. By O. Schreier and E. Sperner. Pp. 238. RM. 9.60. 1931. (Teubner)

Schreier intended eventually to publish his Hamburg lectures on analytical geometry and algebra in book form; and after his premature death in 1929 these were completed and edited by his pupil Sperner. Three-quarters of the present first volume discuss the analytical geometry of n dimensions and the relevant algebra in a unified way; no special knowledge is assumed, but considerable demands are often made on the attention of the reader.

The first chapter deals with affine space and linear equations. A point in affine space of n dimensions is defined as an ordered n-ple of real numbers; lines, intervals and vectors are introduced in the usual analytical fashion, and the vector notation is used fairly freely as a shorthand. Then follows the theory of the existence of solutions of a set of linear equations, the number of equations not being necessarily equal to the number of unknowns; this theory is independent of the use of determinants, and has become fashionable partly because, in a slightly extended form, it can be used in the theory of integral equations. There are places in the treatment here, where a bolder use of vectors, divorced from their coordinates, would have simplified the presenta-

tion. An important theorem which makes the theory very smooth is in recent literature described as Steinitz's Austauschsatz (this book, p. 23; van der Waerden, *Moderne Algebra*, i, p. 96). It ought to be pointed out that the same theorem was given by Grassmann in the second *Ausdehnungslehre* (§ 20, p. 10 of the 1862 edition), and that he uses it in the modern way.

Chapter two, on Euclidean space and determinant theory, defines the distance of two points by the Pythagorean formula, and introduces the cosines of angles by means of the usual cosine formula. The angle concept itself is not actually used, though this is not made clear in the text. Volumes appear in connection with determinants, the latter being defined as functions of vectors which satisfy three conditions. The fundamental theorems on determinants, such as the rules for multiplication, the Laplace expansion, are proved in their general form without mitigation or remorse, but the text is relieved by examples, some even numerical. Geometric intuition is called in to illuminate, though not, of course, to demonstrate, various steps. This first part of the book closes with a classification of orthogonal transformations in two and three dimensions, and a discussion of affine transformations.

The second part, on the theory of fields (Körpertheorie), begins by pointing out that in the earlier portion the basic symbols need not represent real numbers, but may stand for any elements satisfying certain laws of combination. These laws are the postulates which define a "field". The investigation of the general laws of divisibility and of the factoring of polynomials over any field, and the usual introduction to ordinary complex numbers, lead up to an interesting proof of the fundamental theorem of algebra in the ordinary complex field, and with this the volume practically ends.

The remarkable developments in algebra in the last few years, in which the Hamburg school has played a notable part, are not treated at all in this book, but here and there, as in the use of the word "ideal" to denote certain sets of polynomials, and in some of the proofs, their influence is seen.

Practically no references are given to the literature, and it is perhaps rather a pity that a simple inequality should receive such a name as the Cauchy-Schwarz inequality, without any hint of the use in other branches which brought it into prominence.

The promised second volume will deal with matrices, linear transformations, elementary divisors, group theory and projective geometry. H. G. F.

Differential Equations from the Algebraic Standpoint. By J. F. Ritt. Pp. x, 172. $2.50. 1932. American Math. Soc. Colloquium Publications, 14. (American Mathematical Society)

Professor Ritt of Columbia University was invited to lecture on the subject of this monograph in September last at California under the auspices of the American Mathematical Society. Prior to that date the monograph appeared as Number XIV of the Colloquium publications of that Society. Its timely appearance in so convenient a form must have been of great advantage to those of his hearers who were unacquainted with Professor Ritt's recent work.

The standpoint of this monograph was first taken by Ritt himself in a paper "Manifolds of Functions defined by systems of Algebraic Differential Equations", published in October, 1930, in the *Transactions of the American Mathematical Society* (xxxii, 569-598). As it is a standpoint probably unfamiliar to all but a few readers of this *Gazette*, it may perhaps be useful to state the basic theorem and to indicate its purport by an example.

A system of differential forms Σ' is said to *hold* another system Σ if every solution of Σ is a solution of Σ'. A system Σ is defined to be *irreducible* if, when G and H are any two forms such that GH holds Σ, then either G or H holds Σ. The totality of solutions of a system is a manifold, which is said to be reducible or irreducible according as the system is reducible or irreducible.

The basic theorem asserts that *every manifold is composed of a finite number*

of irreducible manifolds, that is, given any system Σ, there exists a finite number of irreducible systems $\Sigma_1, \ldots, \Sigma_s$ such that Σ holds every Σ_i. The coefficients of all the forms are assumed to belong to one definite field of functions.

An example is furnished by the equation

$$y'^2 - 4y = 0,$$

which has the general solutions $y = (x-a)^2$, where a is any constant, and a singular solution $y = 0$. It is reducible in the field of all constants. For of the two forms $y'' - 2$ and y', neither vanishes for all solutions of the equation, whereas the product form $(y'' - 2)y'$ does vanish. In other words, the form $(y'' - 2)\,y'$ holds $y'^2 - 4y$, but neither $y'' - 2$ nor y' alone holds it. Actually, the given equation is equivalent to two irreducible systems of equations, namely,

$$y'^2 - 4y = 0, \quad y'' - 2 = 0;$$

and

$$y'^2 - 4y = 0, \quad y' = 0.$$

The decomposition of a reducible system is based on the differential analogy of Hilbert's theorem on the existence of a finite basis for any infinite system of polynomials.

With the exception of a small amount of material needed for the exposition of the rest, the contents of the monograph are not to be found except in original papers. The American Mathematical Society has performed a valuable service in arranging for this publication in connected form. The matter is interesting and, once certain fundamentals are grasped, not specially difficult to read. The excellent printing was done in Germany. E. L. I.

Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik. By M. J. O. Strutt. Pp. viii, 116. RM. 13.60. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band I, Heft 3. 1932. (Julius Springer)

The book deals with the functions arising from consideration of the wave-equation, $(\nabla^2 + k^2)\,u = 0$, in elliptical and in ellipsoidal coordinates. The former leads to Mathieu functions; the latter, with $k^2 = 0$, to what are usually called Lamé functions. The author calls them Lamé potential functions, and uses the term Lamé wave-functions for the functions arising when $k^2 \neq 0$. (The ordinary differential equations of which the functions are solutions, especially the Mathieu equation, arise also from consideration of several physical problems unconnected with the general wave-equation.)

Adequate accounts of Lamé potential functions have been given elsewhere. Previous connected accounts of Mathieu functions, such as that in the last edition of Whittaker and Watson's *Modern Analysis*, are, however, out of date; and there is no connected account of Lamé wave-functions. There is, then, a gap to be filled, and to some extent the work before us serves this purpose. It is a useful book, both for pure mathematicians interested in the theory of special functions, and for applied mathematicians compelled to use the functions in their researches. It is worthy of consultation by both classes, but it is rather superficial, and there is no pretension to a complete treatise, so that it is doubtful if anyone will find here all that he requires without recourse to further literature. This, too, may be unsuccessful in many cases, for there is a vast amount of unexplored territory. Despite moderately substantial recent advances in the theory and computation of Mathieu functions, much that we should like to know is still hidden, whilst our knowledge of Lamé wave-functions is still exceedingly meagre. Indeed, the account of these functions in this book relates almost entirely to formulae valid only when the k^2 of the wave-equation is small.

Although a fairly representative account of some of the rather pedestrian

methods which have so far proved most fruitful may be found in the brochure, nevertheless there are significant omissions. There is no account of integral equations, and space might profitably have been devoted to a short account of the method recently employed by Whittaker to find recurrence formulae for the Mathieu functions. The applied mathematician should be warned that he will find no tables, since even such tables as have been published have not been reproduced. Again, the zeros of the functions, so important in many physical investigations, have not been discussed, although the zeros of the Mathieu functions, for example, have been studied by Jeffreys, Ince, Hille and Goldstein.

On p. 37 the author quotes the asymptotic formula for the characteristic numbers of the Mathieu equation from my paper in the *Transactions of the Cambridge Philosophical Society*. The last term given was wrong, and the corrected formula was given in a paper of mine in the *Proceedings of the Royal Society of Edinburgh*. The author refers, indeed, to this last paper, but quotes the uncorrected formula nevertheless. Nor does he give the main result of this last paper—an asymptotic expression for the difference of two neighbouring characteristic numbers, use of which I should strongly recommend to anyone who wishes to do any computation in connection with the Mathieu equation.

The book closes with a section on physical applications. The mathematical problems are formulated, and formal solutions are given, but no numerical results. The value of this section is doubtful, since anyone who wishes to study the physical applications could probably formulate the problem and write down the formal solution for himself.

The book is issued in paper covers, and is very expensive.

SYDNEY GOLDSTEIN.

Geometrical Drawing. By G. A. HANBY. Pp. vii, 240. 6s. 1932. (Pitman)

This book is admirably suited to the requirements of the more advanced students in Secondary Schools and Technical Colleges.

The size, L.P. 8vo, enables the subject to be dealt with in clear type and with numerous diagrams none of which appear crowded.

The first part is devoted to examples in Plane Geometry, but considerably more than half the book is given to Solid Geometry. In the early chapters the usual cases are considered for drawing plane figures. They are set out in a simple and direct style, making it easy for the student to follow. No attempt is made to give proofs, since it is essentially a text-book on drawing.

In the chapters dealing with Loci, Conic Sections, and Special Curves the method of plotting the points is clearly demonstrated. Unfortunately, the irregular curves shown detract from the value of the diagrams. A few tortuous curves well drawn would have been appreciated.

The chapters on Solid Geometry in general restrict the application to the usual geometrical solids. There are a few exceptions; for instance, the projection of the helix which is applied to the screw-thread of engineering practice.

Lines and Planes in space are the subjects of an excellent section. The author is to be congratulated upon his treatment and diagrams of oblique planes. The student will benefit by the insistence upon the method of auxiliary views; that is, of changing the oblique plane into the equivalent perpendicular plane before proceeding with the required projection. Teachers of Geometrical Drawing may wonder why the direct method, using the traces, has been relegated to such an unimportant position.

At the end of each chapter throughout this volume there are many well-chosen exercises, and a very useful set of eight test papers has been included at the end, covering the whole of the syllabus.

G. M. CAWLEY.

Mathematical Tables. Vol. II. Emden Functions, being solutions of Emden's Equation together with certain associated Functions. Pp. viii, 34. 7s. 6d. 1932. (Office of the British Association, Burlington House, London.)

This volume contains, to seven decimals, values of $y_n(x)$, $y'_n(x)$, and to fewer places, derivatives up to the sixth order, where $y_n(x)$ satisfies Emden's Equation, namely

$$y'' + 2x^{-1}y + y^n = 0,$$

and the initial conditions $y_n(0) = 1$, $y'_n(0) = 0$. Particular cases are

$$y_1(x) = x^{-1}\sin x, \quad y_5(x) = (1 + \tfrac{1}{3}x^2)^{-\frac{1}{2}}.$$

The values of n used are 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, a separate table being given for each value of n. For each n the argument x increases from 0 to the first zero of $y_n(x)$ except in the case of $y_5(x)$ where tabulation is carried up to the third zero. The solutions have the property that if x_0 be a zero of $y_n(x)$, then $A^{2/(n-1)}y_n(Ax)$ is also a solution which has the zero x_0A^{-1}, A being arbitrary.

The tables are interpolated by Taylor's theorem, sufficient derivates being given for this purpose, as well as a table of Taylor coefficients. Certain invariants required in astronomical applications are also tabulated.

The values for $n = 1\frac{1}{2}$, 2, 3 were found by J. C. P. Miller. The remainder were calculated by D. H. Sadler, who prepared the MS., wrote the useful introduction and saw the volume through the press. The production is excellent, and the volume should be welcomed by those who have occasion to do numerical work with these functions. L. M. M.-T.

Junior Primary Tests in Arithmetic. By A. S. Pratt and E. E. Kitchener. Pp. 128. Without Answers, 1s.; with Answers, 1s. 6d. 1932. (Harrap)

This book is intended for the use of pupils preparing for County Scholarship and other Junior Examinations. Each test paper is conveniently arranged in sections under headings which show the subject-matter, and includes some questions to be worked mentally and others by drawing and measurement. The book is very suitable for its purpose. W. J. D.

Revision Exercises in Mathematics for General School Examinations. By J. A. Crabb. Pp. ix, 115. 2s. 1931. (Macmillan)

Various exercises in Arithmetic, Algebra and Geometry, such as are met with in these examinations, are here arranged under headings which indicate the subject-matter, and the answers are given at the end of the book, which provides within a small compass abundant material for revision and development. W. J. D.

Test Papers in Arithmetic. By W. Smith. Part I, pp. 48, 9d.; Part II, pp. 43, 9d. 1932. (Murray)

There are 100 sets of graduated exercises together with five pages (at the end of Part I) of easy miscellaneous and mental exercises. A certain freshness and vigour in some of the exercises make them interesting to the pupil as well as to the teacher. The use of decimals is unduly postponed to Set LXXVI. Answers to the questions are supplied free to teachers on application to the publisher. W. J. D.

A New Junior Arithmetic. By F. A. J. Rivett. Pp. 144. Without Answers, 2s.; with Answers (16 pp.), 2s. 6d. 1932. (Arnold)

This is a very useful text-book, designed to cover the arithmetical requirements of the first two years of post-primary education. It consists mainly of well-graduated exercises, short test papers, and typical worked examples. Decimals and the metric system are taken early, before vulgar fractions, and freely used thereafter. There is a short Appendix on "Contracted Methods"

(three pages), followed by a table of square roots taken by permission from *Mathematical Tables* by C. V. Durell. W. J. D.

School Mathematics. By L. CROSLAND. Pp. x, 308. 3s. 6d. 1932. (Macmillan)

This is an attempt to provide in one volume a year's course in Arithmetic, Algebra and Geometry suited to the first-year pupils in a secondary school. It contains a vast number of carefully graduated, and suitable, exercises together with many worked examples and explanatory paragraphs expressed in simple language. But the bookwork, which few boys, perhaps, will read, often leaves much to be desired. For instance, on page 10, "$\frac{x}{y}$ means x divided by y"; on page 33 "$\frac{2}{5}$ stands for two-fifths"; on page 55 the rule for finding the product of two arithmetical fractions is derived from the statement "The area of a rectangle is . . . given by length × breadth" (established on page 6 for integers); and on page 112 it is stated that "The rules of arithmetic apply to algebraic fractions". There appears to be no attempt to show the consistency of these statements. On page 148 we read of a polygon that "If all the sides are equal, the figure is a *regular* polygon"; and on page 149 "If the sides of a polygon are equal, the interior angles are equal". The book contains a table of squares, square roots and reciprocals, an index and 36 pages of "answers". W. J. D.

Graduated Exercises in Elementary Mathematics. By E. N. MOZLEY. Teachers' Edition, pp. xv, 78, 2s.; Pupils' Edition, pp. 51, 1s. 3d. 1932. (Bells)

This book is the outcome of ten years' experience of the teaching of elementary mathematics by an officer retired from the Royal Engineers who loves his subject and has analysed a vast number of examination questions set for scholarships to the public schools and for school certificates. He claims to have arranged under suitable headings every type of question necessary for these examinations, and repudiates the suggestion that such a course could be regarded as "cramming" if the examination questions are well chosen. In the hands of a small class of bright boys controlled by a sympathetic teacher the book will be very stimulating. The questions, generally, are interesting and thought-provoking. But for a heterogeneous form of thirty boys it will be of less use to the pupils than to the master.

In the short section devoted to Arithmetic there is but little arithmetical computation, for most of the questions deal with types of arithmetical problems worked as generalised arithmetic with algebraical symbols. This deficiency is to some extent remedied by the inclusion of some five or six pages at the end of the book of test sums in arithmetic for small boys.

The Teachers' Edition contains an introduction, notes on teaching and answers. There will be general agreement with the author when he says, "Never allow a boy to become an automatic machine. He loves to become one, and will often perform its functions quite well. He likes it because, while so engaged, he has not got to think". But does not the author mean "Hardly ever" when he inserts such a question as "What do you mean by 'minus mountain, plus plate'?" on page 21 under the heading "Graphs"? The Teachers' Edition gives the answer to this conundrum: "That if the squared term is minus, the graph looks like a mountain; if plus, the graph looks like a plate". Apparently the graph is that of a quadratic function of x, and when x is large the term involving x^2 dominates the whole function and determines which way the parabola runs off to infinity.

Among many interesting questions there are some which strike one as

peculiar, some that are likely to lead the boy into a mare's nest, and some that require further consideration by the author. For instance:

"If n is any integer (plus or minus) or zero, what is the best way of expressing a positive even number?" Answer, "$2n^2+2$".

"The line joining the centres of two sides of a triangle is *what* and *what*? Prove it."

"Prove that the angle between tangent and chord is equal to the angle in the alternate segment.... Hence prove that the three angles of a triangle add up to 180°."

"(i) Define a logarithm. (ii) Why in our system do not negative numbers have logs? (iii) In what system could negative numbers have logs?"

"What are the two different meanings of π? How are they corrected?"

The Teachers' Edition contains much that is interesting to a mathematical master. There is a breeziness and directness about some of the author's statements which are most exhilarating. "No one should touch the work of teaching promising young mathematicians unless he depends entirely on himself, knows the ropes through and through, and above all has a vivid and almost provocative way of dealing with his subject." Young mathematical masters will read with profit Colonel Mozley's Notes on Teaching though they may not agree with all that he says. He sticks to the old method of subtraction—"Change the sign of the bottom line and add". Why not use throughout the method of complementary addition? Thus, taking one of the author's examples,

$$\begin{array}{r} -5 \\ 2 \\ \hline -7 \end{array}$$

why not—

"2 and I want -7 to make -5", later on abbreviated to "2 and -7"?

W. J. D.

Einführung in die Höhere Mathematik. II. By H. VON MANGOLDT. Sixth edition, revised by K. KNOPP. Pp. xv, 634. RM. 22.50. 1932. (Hirzel)

The general aim of this treatise was indicated in a review of the first volume of the revised edition.* The main topics of the second volume are the differential calculus and its applications to geometry, infinite series and the elements of the theory of functions of a complex variable. The implications of the phrase on the title-page—"Für Studierende und zum Selbststudium"—have not been neglected by the reviser in his task of bringing the book into line with present-day requirements. Difficulties are neither ignored nor sidestepped; they are faced and defeated. It is interesting to note in how few places ease of reading disappears under this honest method. Anyone with a slight equipment of mathematical German will be able to read the greater part of this book with ease; most of us will read it with considerable profit.

There is, of course, no great novelty in the contents. The book begins with the differentiation of functions of one real variable and applications to the mean value theorem, Taylor's theorem and maxima and minima. Similar work is then performed for functions of more than one variable, after an interpolated chapter on the theory of series. Curves and surfaces are carefully defined; conics and quadrics have a chapter; another chapter introduces plane differential geometry—tangents and normals, curvature and envelopes. Two chapters on elementary function-theory conclude the volume; they deal with series of complex terms, the elementary complex functions, the concept of the "regular" or "analytic" function and introductory work on conformal representation.

* *Math. Gazette*, XVI, May, 1932, 158-9.

A few of the many points worthy of notice may here be mentioned. The proof of the formula for the derivative of a function of a function is not the slipshod and dishonest one too common in English text-books; it is rigorous and, though naturally longer than the usual "proof", essentially simple. Newton's rule for obtaining approximations to the roots of an equation is given a particularly careful treatment, though we are sorry that nothing is said about the modifications which can be made to lighten the numerical calculations. Other sections of special merit are those on indeterminate forms, implicit functions, and differentials. Differentials have been frequently discussed in recent *Gazettes*; the treatment given by Knopp is worth looking up. But we venture to add our personal opinion that it is not worth while discussing differentials until functions of more than one variable are considered; for only with such functions does a distinction between having a derivative and being differentiable arise, thereby showing that the latter concept is intrinsically valuable.

It will be obvious that we have a very high opinion of this book; we have enjoyed reading it, and are glad to recommend it very warmly. Having said this, it may be safe to indulge in a small grumble. The chapter on curves and surfaces of the second degree, concerned largely with the reduction of the equation of the second degree to the various standard forms, is conscientious but depressing. The algebra seems to lose elegance, the touch is heavy. Is it that the author has forgotten that, in algebraic geometry, the language may be algebra but the subject must be geometry? T. A. A. B.

The Quadrilateral. An investigation of its chief properties and a synopsis of old theorems and problems. By R. SMYTH. Pp. 42. No price stated. 1931. ("Home Words" Publishing Company, London)

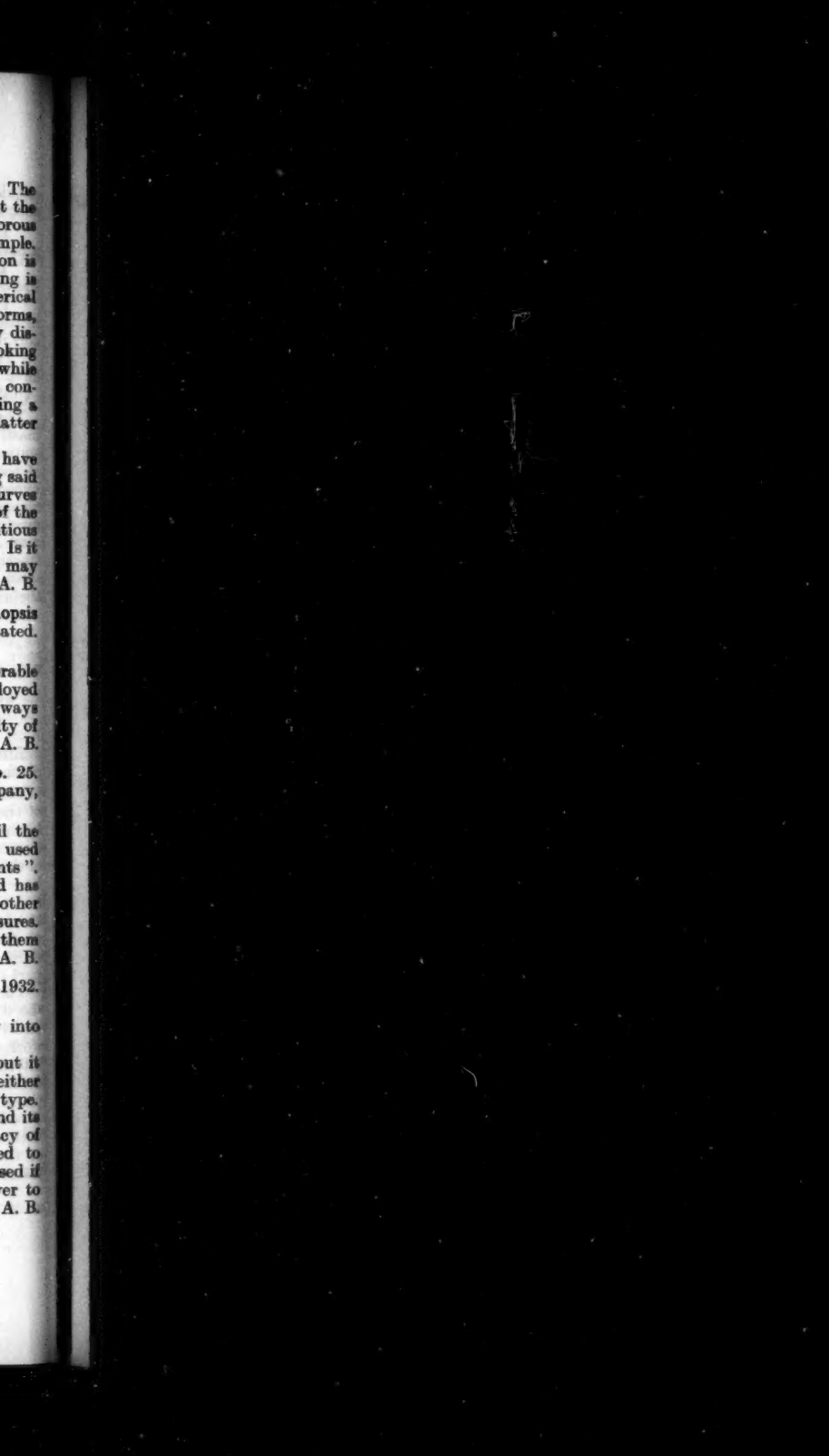
This is a very interesting little compilation, and should be of considerable value to teachers. The exposition is concise—not a single diagram is employed—and there is a collection of thirty examples. The methods are not always purely geometrical, algebra and trigonometry being freely used for brevity of argument. T. A. A. B.

Craftsmen's Measures in Prehistoric Times. By L. M. MANN. Pp. 25. Wrappers, 3s. 6d. Cloth, 5s. No date given. (Mann Publishing Company, Glasgow)

The author's thesis is that "from early in the Old Stone Age until the beginning of the Christian Era, the artificer of every race and country used certain standards of measure for his weapons, implements and ornaments". He asserts the existence, in those times, of two main linear units, and has measured many prehistoric objects to find that they were made to one or other of these scales; he also finds traces of these units in modern standard measures. The coincidences are interesting, but to base a far-reaching induction on them is to take a very big step. T. A. A. B.

Related Mathematics. By W. L. BROWN. Pp. x, 290. 13s. 6d. 1932. (John Wiley, New York; Chapman & Hall, London)

This book is intended for the use of students preparing for entry into industry. It is clearly printed on good paper.

We cannot, of course, presume to teach publishers their business, but it seems absurd to ask 13s. 6d. for a very elementary book of 300 pages. Neither pupils nor teachers can afford such a price even for a useful book of this type. But the absurdity becomes outrageous as soon as the book is opened and its uselessness discovered. The author has a profound belief in the efficacy of dogmatic rules. One example must suffice: four pages are devoted to drumming in the rule that in a simple equation a term may be transposed if at the same time its sign is changed, but there is no attempt whatever to show any reason for this step. T. A. A. B.

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